NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

MA3005 Control Theory

Tutorial 8

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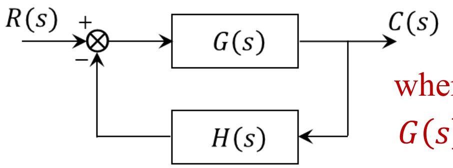
Root Locus



where

$$G(s)H(s) = K\frac{N(s)}{D(s)}$$

CE: $1 + G(s)H(s) = 0$

$$1 + K\frac{N(s)}{D(s)} = 0$$

$\Rightarrow$ As K varies, pole position varies!

Root locus is the trajectory of the CL poles as
$K = 0 \rightarrow \infty$

All roots s_0 should satisfy this equation:

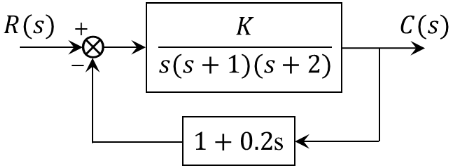
$$G(s)H(s) = -1$$

$$K\frac{N(s)}{D(s)} = 1e^{-j\pi}$$

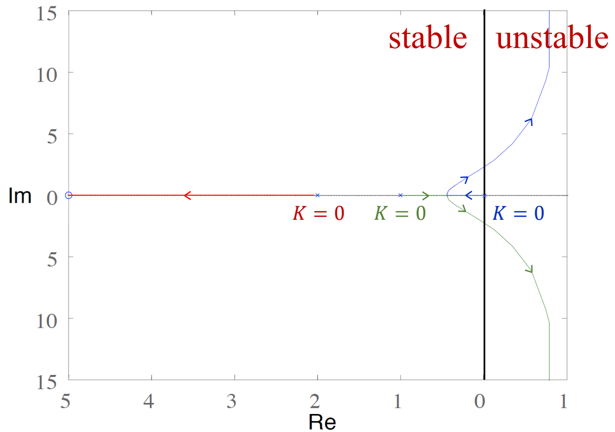
Magnitude condition $\Rightarrow K\left|\frac{N(s_0)}{D(s_0)}\right| = 1$

Angle condition $\Rightarrow \angle N(s_0) - \angle D(s_0) = -\pi(1+2q)$

Example:

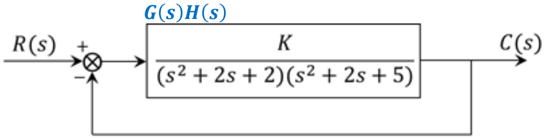


Root locus



Question 1

1. Determine the starting and ending points.
2. Show that the departing angles from the top two starting points are respectively $\frac{-\pi}{2}$ and $\frac{\pi}{2}$
3. Determine the break-in/out points
4. Sketch the asymptotes
5. Find the intersection of the root locus with the imaginary axis and the corresponding gain K



1.

$$G(s)H(s) = \frac{K}{(s^2+2s+2)(s^2+2s+5)}, \qquad H(s) = 1$$

Closed loop CE:

(Proof)

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{(s^2+2s+2)(s^2+2s+5)} = 0$$

$$(s^2+2s+2)(s^2+2s+5) + K = 0$$

For starting points ($K = 0$):

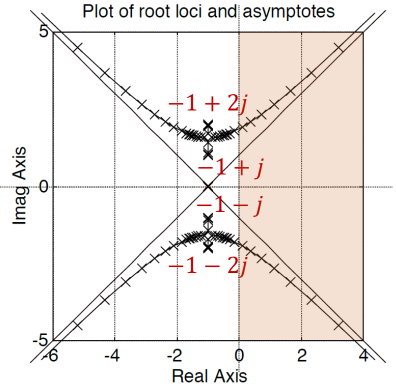
$$(s^2+2s+2)(s^2+2s+5) = 0$$

Starting points are the open loop poles:

$$s = -1 \pm j \qquad s = -1 \pm j2$$

There are no finite zeros ⇒ Ending points must be at infinity

Observe that there are no real roots for any values of K



- Starting points ($K = 0$) are poles of open loop GH
- Ending points ($K = \infty$) are finite GH zeros or infinite zeros
- For infinite zeros: end directions to ∞ given by Asymptotes

Question 1

$$G(s)H(s) = K\frac{N(s)}{D(s)} = K\frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

2. Show that the departing angles from the top two starting points are respectively $\frac{-\pi}{2}$ and $\frac{\pi}{2}$

Open loop poles (starting points): $-1 \pm j\ ,\ -1 \pm j2$

Departure angle:

$$\sum_{i=1}^{m} \angle(s+z_i) - \sum_{j=1}^{n} \angle(s+p_j) = -\pi$$

For $-1+j$:

(no finite zeros)

Take note of the approximation in the angles

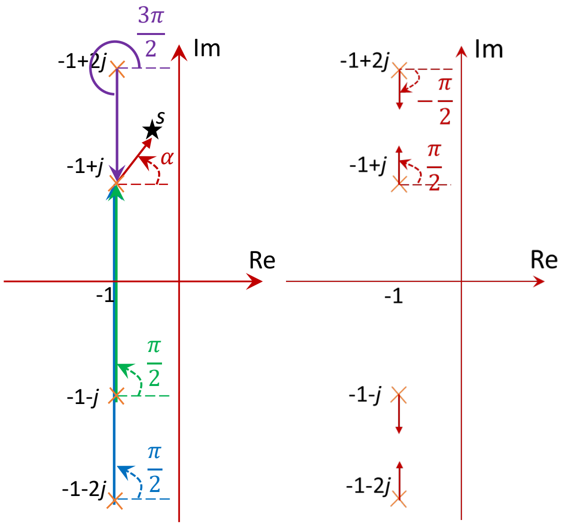
$$\Rightarrow 0 - \sum_{j=1}^{n=4} \angle(s+p_j) = -\left(\alpha + \frac{\pi}{2} + \frac{\pi}{2} + \frac{3\pi}{2}\right) = -\pi \qquad \Rightarrow \alpha = -\frac{3\pi}{2} \text{ or } \frac{\pi}{2}$$

$(90°)$

For $-1+2j$:

$$\Rightarrow 0 - \sum_{j=1}^{n=4} \angle(s+p_j) = -\left(\alpha + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}\right) = -\pi \qquad \Rightarrow \alpha = -\frac{\pi}{2}$$

$(-90°)$



Root Locus symmetrical about the real axis

Question 1

3. Determine the break-in/out points

$$\boxed{\text{Break-out and break-in points: } \frac{dK}{ds} = 0}$$

Need to check if result is valid

Between 2 open loop poles, the break-out points are given by:

$$\text{CE: } (s^2 + 2s + 2)(s^2 + 2s + 5) + K = 0$$

$$K = -(s^2 + 2s + 2)(s^2 + 2s + 5)$$

$$\frac{dK}{ds} = -(2s + 2)(s^2 + 2s + 5) - (2s + 2)(s^2 + 2s + 2) = 0$$

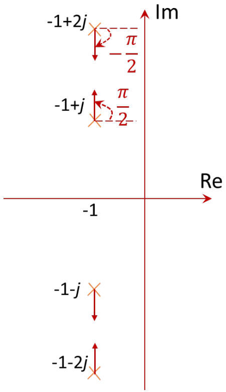
$$2(s + 1)(2s^2 + 4s + 7) = 0$$

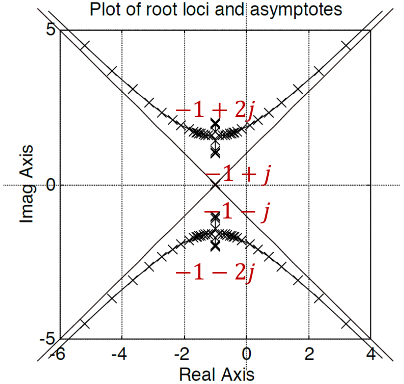
$s = -1$ or $-1 \pm j1.58$

In between the open loop poles

$\Rightarrow$ not on locus & $K = -4$ (rejected)

$\Rightarrow$ Break-out points





Question 1

$$G(s)H(s) = K\frac{N(s)}{D(s)} = K\frac{(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)}$$

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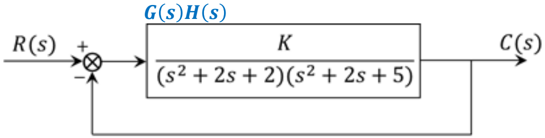
4. Sketch the asymptotes



No finite zeros ⇒ Ending points must be at infinity
⇒ There are 4 asymptotes

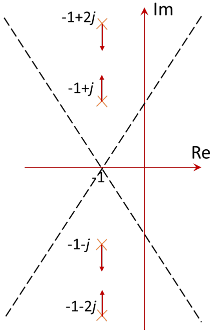
Asymptote angle:

$$\angle s = \frac{\pi(1+2q)}{[n-m]} \quad , q = 0, \dots, n-m-1$$

$$q = 0, \dots, 3$$

$$\angle s = \frac{\pi(1+2q)}{4-0} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$$

$(\pm 45°, \pm 135°)$



Real axis intercept:

$$\sigma_a = \frac{\sum_i^n(-p_i) - \sum_j^m(-z_j)}{n-m} = \frac{\sum GH \text{ finite poles} - \sum GH \text{ finite zeros}}{n-m}$$

$$\sigma_a = \frac{\sum_i^n (-1+j)+(-1-j)+(-1+2j)+(-1-2j)}{4} = \frac{-4}{4} = -1$$

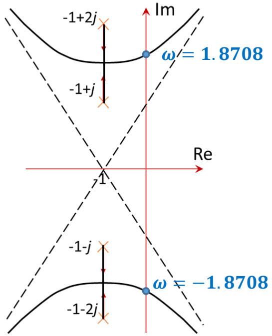
Note that imaginary parts will cancel off

Question 1

5. Find the intersection of the root locus with the imaginary axis and the corresponding gain K

Crossing of imaginary axis:

Set $s = j\omega$ on the CE, then solve for the unknowns



CE: $$1 + \frac{K}{(s^2+2s+2)(s^2+2s+5)} = 0$$

$$(s^2+2s+2)(s^2+2s+5) + K = 0$$

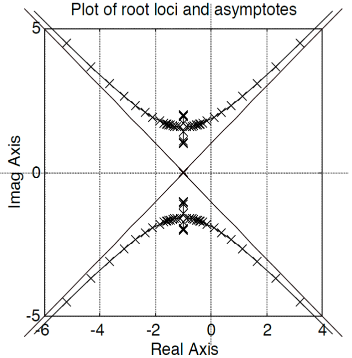
$$((j\omega)^2+2(j\omega)+2)((j\omega)^2+2(j\omega)+5) + K = 0$$

$$(\omega^4 - 11\omega^2 + 10 + K) + j(-4\omega^3 + 14\omega) = 0$$

$$\Rightarrow \omega^4 - 11\omega^2 + 10 + K = 0 \qquad \Rightarrow -4\omega^3 + 14\omega = 0$$

$$K = -(1.8708)^4 + 11(1.8708)^2 - 10 \qquad \omega = \pm 1.8708$$

$$K = 16.2495$$



Root Locus Pointers

1. Starting and ending points of the locus
   a. Starting points – open-loop GH poles
   b. Ending points – open-loop GH zeros

The number of branches equals to the order of CE
Ending points ($K = \infty$) are finite GH or infinite zeros
Root Locus symmetrical about the real axis

2. Locus on real axis

Segments on the real axis to the left of odd number real-axis open-loop poles and zeros are parts of root locus.

3. Break-in and break-out points
   a. Break-in (minimum K)
   b. Break-out (maximum K)

$$\frac{dK}{ds} = 0$$

4. Imaginary axis intersection

Set $s = j\omega$ in CE

5. Asymptotes (infinite open-loop zeros)
   a. Angles
   b. Location

Asymptote angle:

$$\angle s = \frac{\pi(1+2q)}{[n-m]}, q = 0, \dots, n-m-1$$

Real axis intercept:

$$\sigma_a = \frac{\sum GH \text{ finite poles} - \sum GH \text{ finite zeros}}{n-m}$$

6. Departure and arrival angles (for complex poles/zeros)

Departure angle:

$$\sum_{i=1}^{m} \angle(s+z_i) - \sum_{j=1}^{n} \angle(s+p_j) = -\pi$$

Arrival angle:

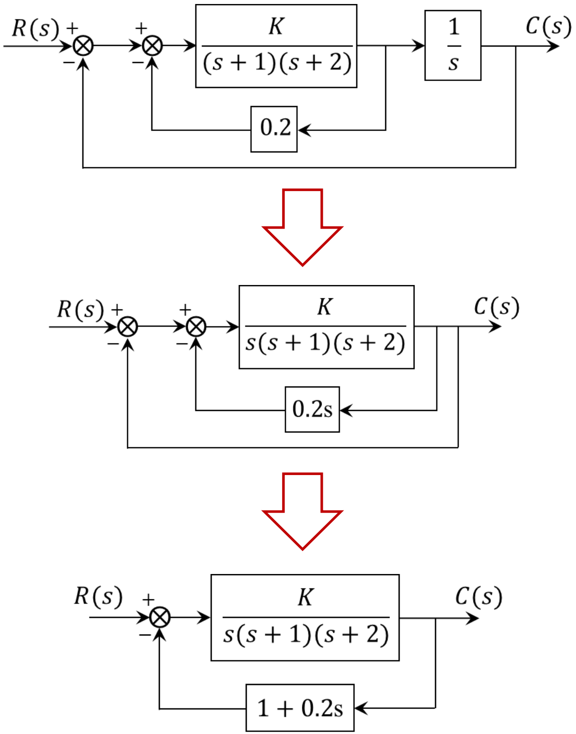
$$\sum_{i=1}^{m} \angle(s+z_i) - \sum_{j=1}^{n} \angle(s+p_j) = \pi$$

Question 2

Consider the following system with variable gain K

i. Sketch the root locus of the system (detail every steps).

ii. With the help of root locus, explain the effects of increasing K on the system's transient behavior.



i.

1. Starting and ending points of the locus:

$$\text{CE: } 1 + G(s)H(s) = 0$$

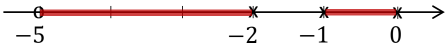
$$1 + K\frac{(1 + 0.2s)}{s(s+1)(s+2)} = 0$$

Starting points: $s = 0, s = -1, s = -2$

Ending points: $s = -5$ (finite zero) and two infinite zeros

2. Locus of real axis:

Segments on the real axis to the left of odd number real-axis open-loop poles and zeros are parts of root locus.



One real axis segment between 0 and –1, and the other two segments between – 2 and –5 will break out some where between them.

Question 2

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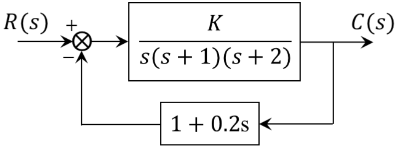
3. Break-in and break-out points:



CE: $1 + G(s)H(s) = 1 + K\dfrac{(1+0.2s)}{s(s+1)(s+2)} = 0$

$$\Rightarrow K = \frac{-s(s+1)(s+2)}{(1+0.2s)}$$

$$\frac{dK}{ds} = -\frac{(1+0.2s)\left(3s^2+6s+2\right) - s(s+1)(s+2)(0.2)}{(1+0.2s)^2} = 0$$

$$(1+0.2s)\left(3s^2+6s+2\right) - s(s+1)(s+2)(0.2) = 0$$

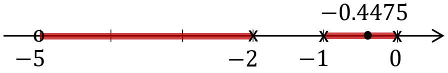
$$(5+s)\left(3s^2+6s+2\right) - s(s+1)(s+2) = 0$$

$$\left(3s^3+21s^2+32s+10\right) - \left(s^3+3s^2+2s\right) = 0$$

$$2s^3+18s^2+30s+10 = 0$$

$$s = -6.934, -1.6091, -0.4475$$

(rejected) (rejected)



4. Imaginary axis intersection:

Set $s = j\omega$ in CE

$$j\omega(j\omega+1)(j\omega+2) + K(1+0.2j\omega) = 0$$

$$\left(-3\omega^2+K\right) + j\left(-\omega^3+(2+0.2K)\omega\right) = 0$$

$\Rightarrow -3\omega^2 + K = 0$ $\qquad$ $\Rightarrow -\omega^3 + (2+0.2K)\omega = 0$

$-3(2+0.2K) + K = 0$ $\qquad$ $-\omega^2 + (2+0.2K) = 0$

$-6 - 0.6K + K = 0$ $\qquad$ $\omega^2 = 2 + 0.2K$

$-6 - 0.6K + K = 0$

$K = 15$ $\qquad$ $\omega^2 = 2 + 0.2(15)$

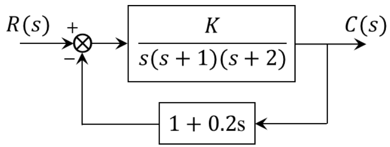
$\omega = \sqrt{5}$

Question 2

5. Asymptotes:

CE: $s(s+1)(s+2) + K(1+0.2s) = 0$

$$G(s)H(s) = \frac{K(1+0.2s)}{s(s+1)(s+2)}$$



Angles:

$$\angle s = \frac{\pi(1+2q)}{[n-m]} \quad , q = 0, \dots, n-m-1$$

$$= \frac{\pi(1+2q)}{[3-1]} \quad , q = 0,1$$

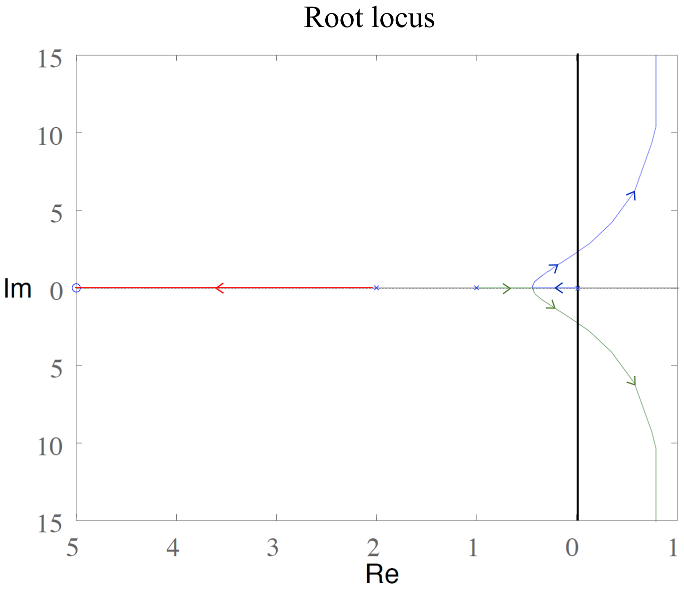
$$= \frac{\pi}{2}, \frac{3\pi}{2}$$

Real axis intercept:

$$\sigma_a = \frac{\sum GH \text{ finite poles} - \sum GH \text{ finite zeros}}{n-m}$$

$$= \frac{0-1-2-(-5)}{2} = 1$$

This means that part of the locus will cross over the RHP



Question 2

ii. With the help of root locus, explain the effects of increasing K on the system's transient behavior.

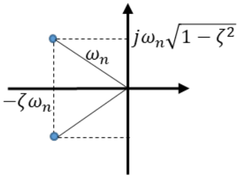
Let's say:

For $K = 1.3$, CE: $s(s+1)(s+2) + K(1+0.2s) = 0$

$$s^3 + 3s^2 + 2.26s + 1.3 = 0$$

$$s_3 = -2.2530, \quad s_{1,2} = -0.3735 \pm j0.6614$$

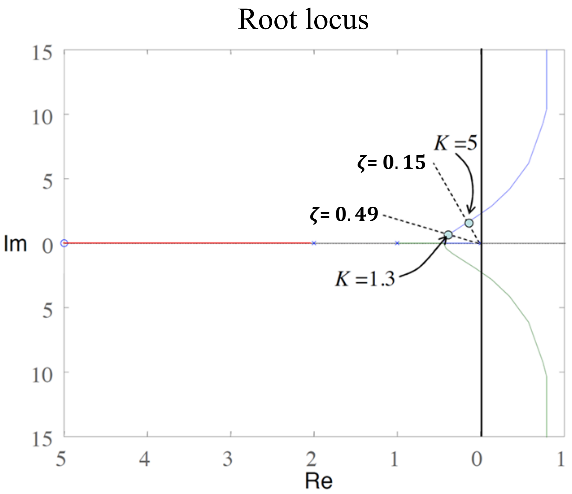
$$(\zeta = 0.49)$$



For $K = 5$, CE: $s^3 + 3s^2 + 3s + 5 = 0$

$$s_3 = -2.5874, \quad s_{1,2} = -0.2063 \pm j1.3747$$

$$(\zeta = 0.15)$$



Increasing K from $K = 1.3$ will reduce the damping from $\zeta = 0.49$ to $\zeta = 0.15$ and the resulting time response will be more oscillatory

Beyond $K = 15$, the root locus crosses over to the right half plane, indicating the system become unstable ⇒ Divergent oscillation