NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

MA3005 Control Theory

Tutorial 7

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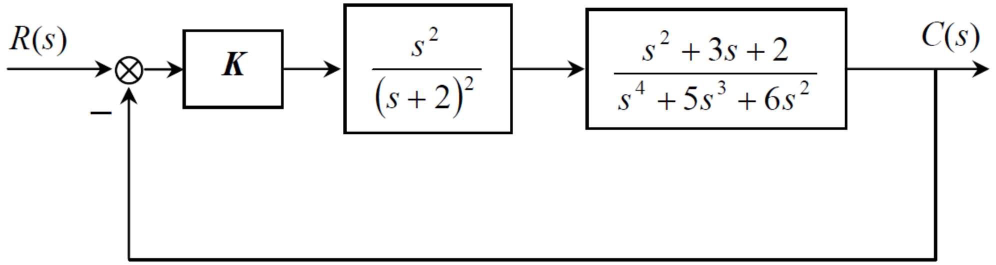
Question 1

1. What is the system type? Compute the static position error constant K_p of the system.
2. The DC gain of a system with transfer function G is defined as $\lim_{s\to 0} G(s)$. Compute the DC gain of the open-loop and the closed-loop system. Explain the name "DC gain".
3. What is the steady-state error of this system in response to a unit-step input? To unit-ramp input?



G(s): Open-loop transfer function for unity feedback system

$$G(s) = \frac{K(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{\boxed{s^N}(T_1 s+1)(T_2 s+1)\cdots(T_p s+1)}$$

G(s) is Type *N* system,
where *N* can be 0, 1, 2, etc..

1.

$$G(s) = K\frac{s^2(s^2+3s+2)}{(s+2)^2\,(s^4+5s^3+6s^2)} = K\frac{s^2(s+1)(s+2)}{(s+2)^2 s^2(s+3)(s+2)}$$

$$= K\frac{(s+1)}{(s+2)^2\,(s+3)}$$

No integrators $\Rightarrow N = 0 \Rightarrow$ System type 0

$$K_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} K\frac{(s+1)}{(s+2)^2\,(s+3)} = \frac{K}{12}$$

Question 1

2. The DC gain of a system with transfer function G is defined as $\lim_{s\to 0} G(s)$. Compute the DC gain of the open-loop and the closed-loop system. Explain the name "DC gain".
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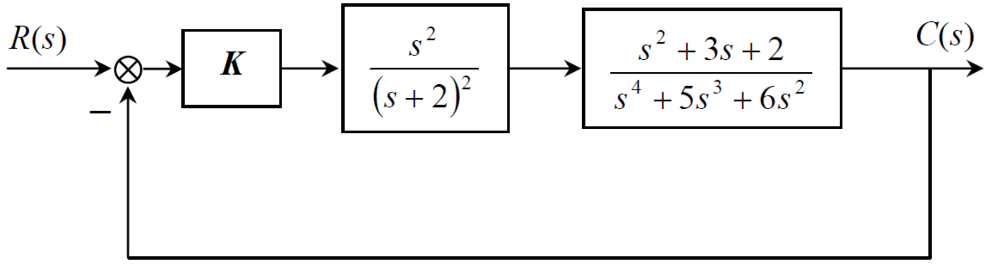


2. Open loop system:

$$\text{DC gain } = \lim_{s\to 0} G(s) = \frac{K}{12}$$

Closed loop system:

$$\frac{C(s)}{R(s)} = \frac{K\frac{(s+1)}{(s+2)^2\,(s+3)}}{1+K\frac{(s+1)}{(s+2)^2\,(s+3)}} = \frac{K(s+1)}{(s+2)^2\,(s+3)+K(s+1)} = \frac{K(s+1)}{s^3+7s^2+(16+K)s+(12+K)}$$

$$\text{DC gain } = \lim_{s\to 0} \frac{C(s)}{R(s)} = \frac{K}{12+K} \quad \rightarrow 1 \text{ when } K \rightarrow \infty$$

Note: Increasing K, increases CL steady state value and hence reduce e_{ss}

DC gain: Steady-state response to a unit step input

$$\lim_{t\to\infty} c_{step}(t) = \lim_{s\to 0} s\,C(s) \qquad C(s) = G(s)R(s)$$

$$= \lim_{s\to 0} s\left(G(s)\,\frac{1}{s}\right)$$

$$= \lim_{s\to 0} G(s)$$

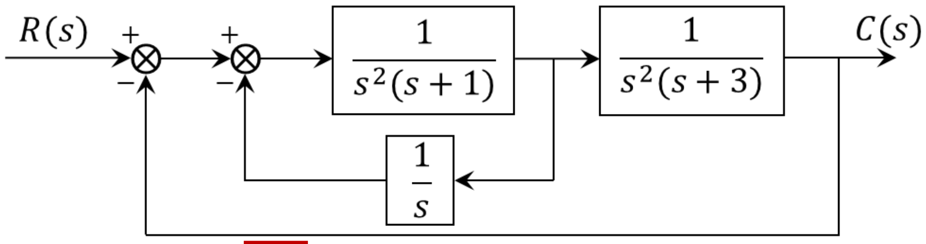
(Week 4 page 19)

3. Step input $e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\frac{K}{12}} = \frac{12}{12+K}$

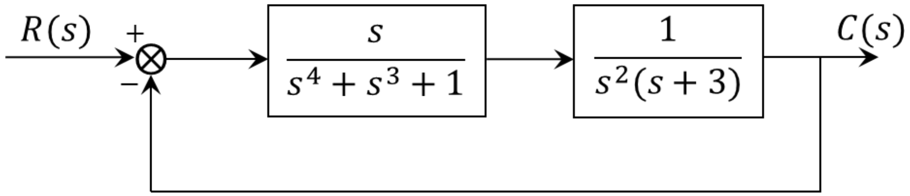
Ramp input $e_{ss} = \frac{1}{K_v} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{\lim_{s\to 0} sK\frac{(s+1)}{(s+2)^2\,(s+3)}} \rightarrow \infty$

Question 2

1. Calculate the closed-loop transfer function and the system type.
2. Compute the steady-state error for the input $r(t) = 5u(t)$ and for the input $r(t) = 5tu(t)$, where $u(t)$ is the unit-step function. Discuss the validity of your answers.



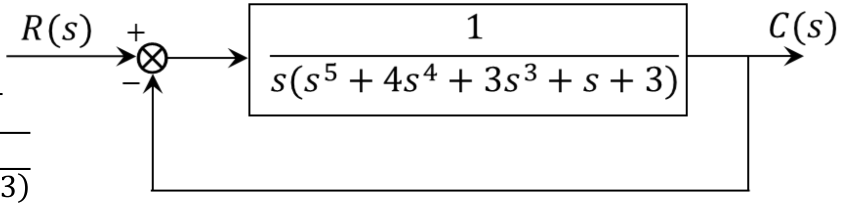
$$TF = \frac{\frac{1}{s^2(s+1)}}{1+\frac{1}{s^3(s+1)}} = \frac{\frac{1}{s^2(s+1)}}{\frac{s^3(s+1)+1}{s^3(s+1)}} = \frac{s}{s^4+s^3+1}$$



1.

$$G_{CL}(s) = \frac{1}{s^6+4s^5+3s^4+s^2+3s+1}$$

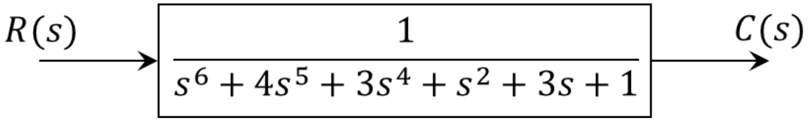
$$G_{OL}(s) = \frac{1}{s(s^5+4s^4+3s^3+s+3)}$$



$$G_{CL}(s) = \frac{\frac{1}{s(s^5+4s^4+3s^3+s+3)}}{1+\frac{1}{s(s^5+4s^4+3s^3+s+3)}}$$

$$= \frac{\frac{1}{s(s^5+4s^4+3s^3+s+3)}}{\frac{s(s^5+4s^4+3s^3+s+3)+1}{s(s^5+4s^4+3s^3+s+3)}}$$

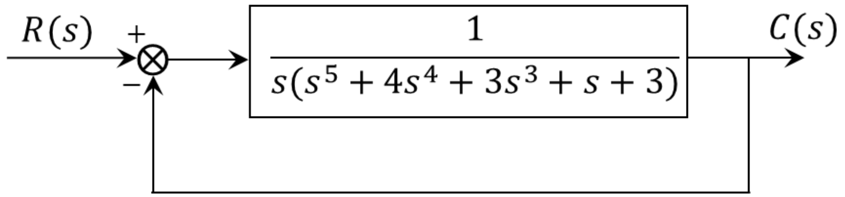
$$= \frac{1}{s^6+4s^5+3s^4+s^2+3s+1}$$



1 integrator $\Rightarrow N = 1 \Rightarrow$ System type 1

Question 2

1. Calculate the closed-loop transfer function and the system type.
2. Compute the steady-state error for the input $r(t) = 5u(t)$ and for the input $r(t) = 5tu(t)$, where $u(t)$ is the unit-step function. Discuss the validity of your answers.



2. $$G_{OL}(s) = \frac{1}{s(s^5 + 4s^4 + 3s^3 + s + 3)}$$

Input $r(t) = 5u(t) \Rightarrow$ step input

For system type 1:

$$\text{Step input } e_{ss} = \frac{\mathbf{5}}{1+K_p} = \frac{5}{1+\lim_{s\to 0} G_{OL}(s)} = 0 \text{ (expected)}$$

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{(1+G(s))}$$

(Week 8 page 19)

$$R(s) = \frac{5}{s} \rightarrow e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \cdot \frac{5}{s} = \frac{5}{1+G(s=0)}$$

If $K_p = \lim_{s\to 0} G(s) = G(s=0)$

$$e_{ss} = \frac{\mathbf{5}}{1+K_p}$$

Input $r(t) = 5tu(t) \Rightarrow$ ramp input

For system type 1:

$$\text{Ramp input } e_{ss} = \frac{\mathbf{5}}{K_v} = \frac{5}{\lim_{s\to 0} sG_{OL}(s)}$$

$$= \frac{5}{\lim_{s\to 0} \dfrac{s}{s(s^5 + 4s^4 + 3s^3 + s + 3)}} = 15$$

Question 2

1. Calculate the closed-loop transfer function and the system type.
2. Compute the steady-state error for the input $r(t) = 5u(t)$ and for the input $r(t) = 5tu(t)$, where $u(t)$ is the unit-step function. Discuss the validity of your answers.

$$R(s) \longrightarrow \boxed{\frac{1}{s^6 + 4s^5 + 3s^4 + s^2 + 3s + 1}} \longrightarrow C(s)$$

2. Check stability using Routh Hurwitz method:

CE: $s^6 + 4s^5 + 3s^4 + s^2 + 3s + 1 = 0$

Routh's Array:

s^6	1	3	1	1
s^5	4	0	3	
s^4	$\frac{12-0}{4} = 3$	$\frac{4-3}{4} = \frac{1}{4}$	$\frac{4-0}{4} = 1$	
s^3	$\frac{0-1}{3} = \frac{-1}{3}$	$\frac{9-4}{3} = \frac{5}{3}$		
s^2	$\frac{\frac{-1}{12}-5}{\frac{-1}{3}} = \frac{61}{4}$	$\frac{\frac{-1}{3}-0}{\frac{-1}{3}} = 1$		
s	$\frac{\frac{61}{4}\left(\frac{5}{3}\right)-\frac{-1}{3}}{\frac{61}{4}} = 1.69$			
s^0	$\frac{1.69-0}{1.69} = 1$			

Necessary (but not sufficient) condition

If a system is stable, then all coefficients of its CE are real, are of the same sign, and none of its coefficients is zero.

Due to missing s^3 coefficient, system can at best marginally stable

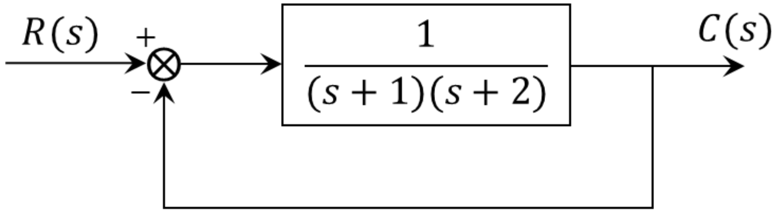
First column:
- If there is no changing of sign ⇒ stable.
- No. of sign changes ⇒ no. of poles in RHP

Since there is 2 sign changes in the first column, therefore there are 2 RHP poles and the system is **unstable**.

The e_{ss} results are meaningless.

Question 3

1. Compute the poles of the system.
2. Compute the steady-state error to unit-step input.
3. To eliminate the steady-state error to unit-step input, one inserts an integral controller $K(s) = \frac{1}{s}$ in the feedforward loop. Compute the poles of the system with integral controller. How does integral action influence the stability of the system?



1.
$$G_{CL}(s) = \frac{\frac{1}{(s+1)(s+2)}}{1 + \frac{1}{(s+1)(s+2)}} = \frac{\frac{1}{(s+1)(s+2)}}{\frac{(s+1)(s+2)+1}{(s+1)(s+2)}} = \frac{1}{s^2 + 3s + 3}$$

CE: $s^2 + 3s + 3 = 0$

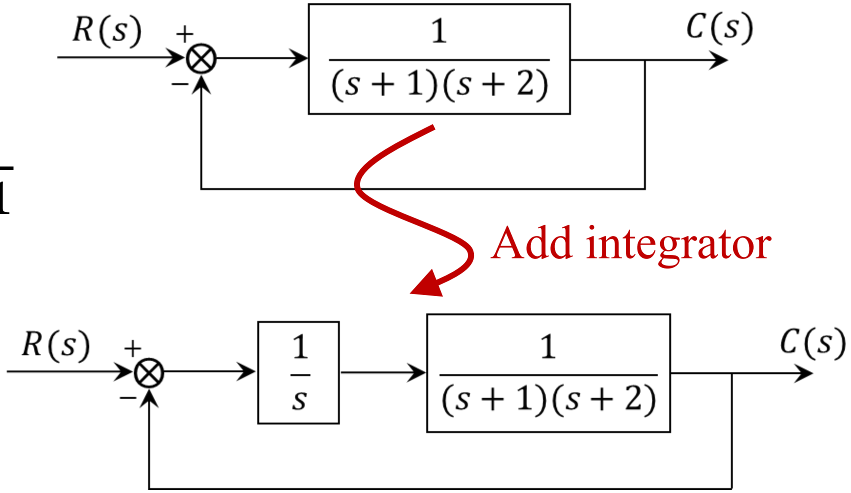
$$s_{1,2} = \frac{-3 \pm j\sqrt{3}}{2} = -1.5 \pm 0.87j$$

2. System type 0:

Step input $e_{SS} = \frac{1}{1 + K_p} = \frac{1}{1 + \lim_{s\to 0} G_{OL}(s)} = \frac{1}{1 + \lim_{s\to 0} \frac{1}{(s+1)(s+2)}} = \frac{2}{3}$

Question 3

3. To eliminate the steady-state error to unit-step input, one inserts an integral controller $K(s) = \frac{1}{s}$ in the feedforward loop. Compute the poles of the system with integral controller. How does integral action influence the stability of the system?



3.

$$G_{NEW}(s) = \frac{\frac{1}{s(s+1)(s+2)}}{1+\frac{1}{s(s+1)(s+2)}} = \frac{\frac{1}{s(s+1)(s+2)}}{\frac{s(s+1)(s+2)+1}{s(s+1)(s+2)}} = \frac{1}{s^3+3s^2+2s+1}$$

CE: $s^3 + 3s^2 + 2s + 1 = 0$

$s_1 = -2.32 \qquad s_{2,3} = -0.34 \pm 0.56j$

$$G_{OLD}(s) = \frac{1}{s^2+3s+3}$$

$$s_{1,2} = \frac{-3 \pm j\sqrt{3}}{2} = -1.5 \pm 0.87j$$

The dominant roots of the new system are on the right of the dominant roots of the original system. Thus, the new system is less stable (but still stable)