NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

MA3005 Control Theory

Tutorial 6

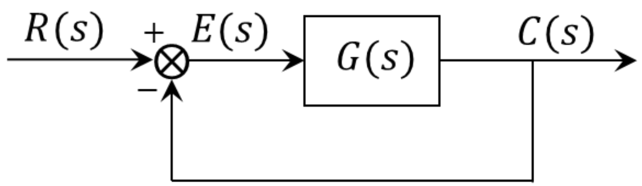
Asst. Prof. Mir Feroskhan

Question 1

Consider a unity-feedback control system whose open-loop transfer function is $G(s) = \dfrac{K}{s(Js+B)}$

1. Suppose that the system is subject to unit-ramp input. Discuss how the values of K and B affect the steady-state error.
2. Sketch typical unit-ramp response curves for a small value, a medium and a large value of K.

1.



$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{s(Js+B)}}{1+\dfrac{K}{s(Js+B)}} = \frac{\dfrac{K}{s(Js+B)}}{\dfrac{Js^2+Bs+K}{s(Js+B)}} = \frac{K}{Js^2+Bs+K}$$

$$E(s) = R(s) - C(s) = \frac{1}{s^2} - \frac{K}{Js^2+Bs+K}\cdot\frac{1}{s^2} = \frac{Js+B}{Js^3+Bs^2+Ks}$$

$$e_{ss} = e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{Js^2+Bs}{Js^3+Bs^2+Ks} = \frac{B}{K}$$

It can be deduced that we can reduce the steady state error, e_{ss} by either increasing the gain K or decreasing B.

Only applicable for unity-feedback system

Alternative method:

$$G(s) = \frac{\dfrac{K}{B}}{s\left(\dfrac{J}{B}s+1\right)}$$

Type 1 system with ramp input:

$$e_{ss} = \frac{1}{K_v} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{\dfrac{K}{B}} = \frac{B}{K}$$

G(s): Open-loop transfer function for unity feedback system

$$G(s) = \frac{K(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{\boxed{s^N}(T_1 s+1)(T_2 s+1)\cdots(T_p s+1)}$$

G(s) is Type *N* system,
where *N* can be 0, 1, 2, etc..

Question 1

Examining the closed loop TF further:

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K} = \frac{\frac{K}{J}}{s^2 + \frac{B}{J}s + \frac{K}{J}} = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

$${\omega_n}^2 = \frac{K}{J} \quad \Rightarrow \quad \omega_n \propto \sqrt{K}$$

$$2\zeta\omega_n = \frac{B}{J} \quad \Rightarrow \quad \zeta = \frac{B}{2J\omega_n} \quad \Rightarrow \quad \zeta \propto \frac{1}{\sqrt{K}}$$

$$\Rightarrow \quad \zeta \propto B$$

By increasing K or decreasing B, the damping ratio (ζ) is reduced, resulting in a more oscillatory transient response.

If increase K:

Doubling K in $e_{SS} = \frac{B}{K}$ will reduce e_{SS} by half, but will reduce ζ to 0.707 of its original value too.

If decrease B:

Halving B will reduce both e_{SS} and ζ by half (lower than 0.707$\Rightarrow$ more oscillatory).

Therefore, it is better to increase K to reduce e_{SS} than to decrease B.

Question 1

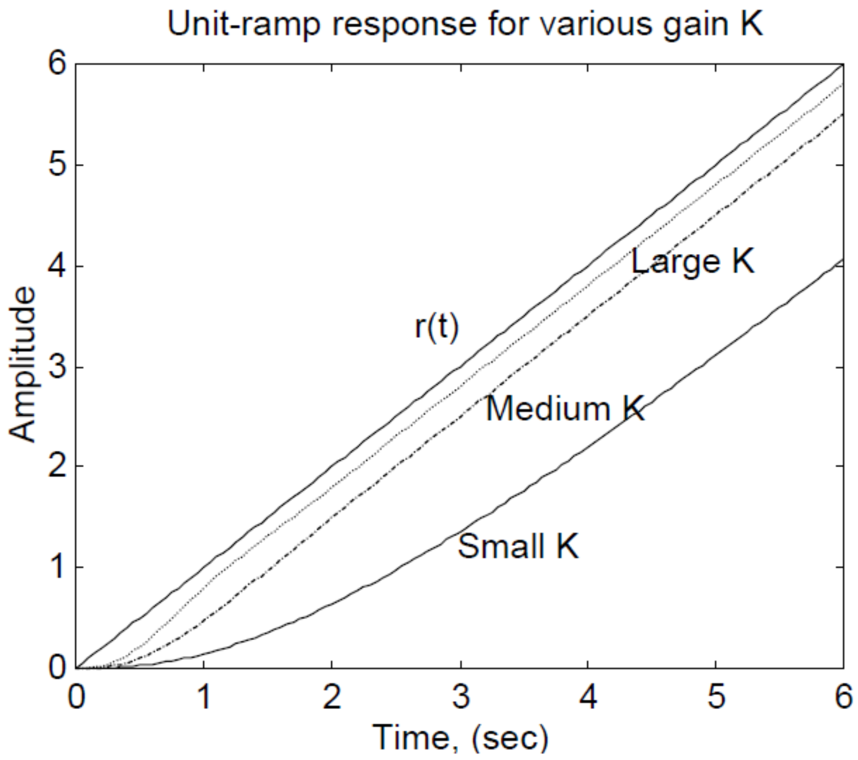
2. Sketch typical unit-ramp response curves for a small value, a medium and a large value of K.

Choose $J = 1$, $B = 4$, and $K = 2$, 8 and 20.
As shown, the output is able to track the input ($R(s) = \frac{1}{s^2} \Rightarrow r(t) = t$) at steady state.
But there is a steady state error, $e_{ss} = \frac{B}{K}$.
Note that at large K, the transient response is more oscillatory as was predicted.

$$G(s) = \frac{\frac{K}{B}}{s\left(\frac{J}{B}s + 1\right)}$$

Type 1 system with ramp input:

$$e_{ss} = \frac{1}{K_v} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{B}{K}$$



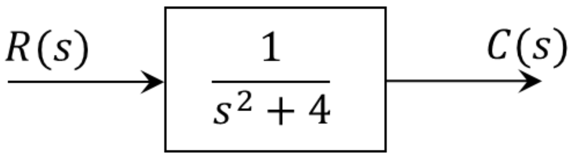
MATLAB script: $\frac{C(s)}{R(s)} = \frac{K}{Js^2+Bs+K}$

```
J=1;B=4;K1=2;K2=8;K3=20;

num1=K1;den1=[J B K1 0];
num2=K2;den2=[J B K2 0];
num3=K3;den3=[J B K3 0];
sys1=tf(num1,den1);
sys2=tf(num2,den2);
sys3=tf(num3,den3);

time=0:0.1:10;
y1=step(sys1,time);
y2=step(sys2,time);
y3=step(sys3,time);

figure
plot(time,time,time,y1,time,y2,time,y3);
```

Question 2

Consider a second-order system with the transfer function $G(s) = \dfrac{1}{s^2+4}$

1. Calculate and sketch the time response of the system to unit-step input (use Laplace inverse transform).
2. What is the frequency of oscillation of the response?



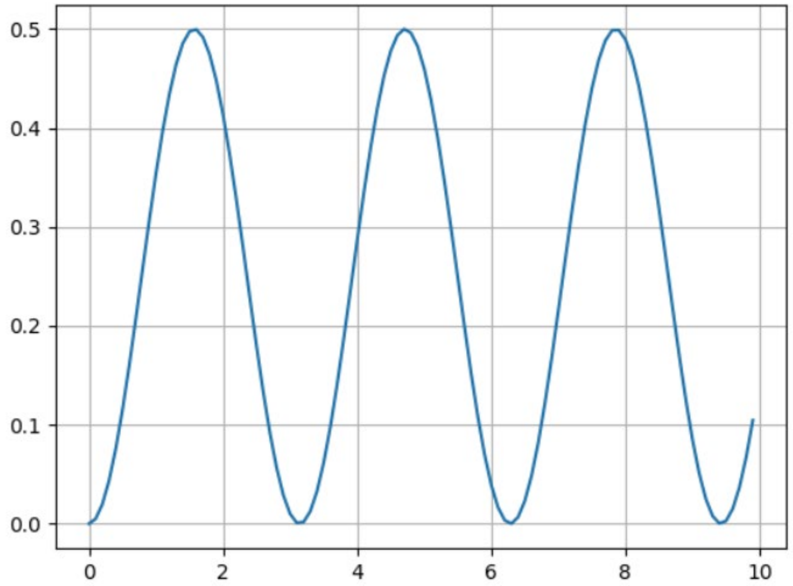
1. $R(s) = \dfrac{1}{s} \quad \Rightarrow \quad C(s) = \dfrac{1}{s^2+4} \cdot \dfrac{1}{s} = \dfrac{1}{s(s-2j)(s+2j)}$

Using partial fractions: $\dfrac{1}{s(s-2j)(s+2j)} = \dfrac{A}{s} + \dfrac{B}{(s-2j)} + \dfrac{C}{(s+2j)}$

$$\Rightarrow 1 = A(s-2j)(s+2j) + Bs(s+2j) + Cs(s-2j)$$

Compare coefficients,

$$\left.\begin{array}{ll} s^2: & A+B+C=0 \\ s^1: & B-C=0 \\ s^0: & 4A=1 \end{array}\right\} \quad A = \frac{1}{4} \qquad B = C = -\frac{1}{8}$$



$$C(s) = \frac{1}{4s} - \frac{1}{8(s-2j)} - \frac{1}{8(s+2j)}$$

$\dfrac{e^{i\theta}+e^{-i\theta}}{2} = \cos\theta$

$$c(t) = \frac{1}{4} - \frac{1}{8}\left(e^{j2t} + e^{-j2t}\right) = \frac{1}{4} - \frac{1}{4}\cos 2t$$

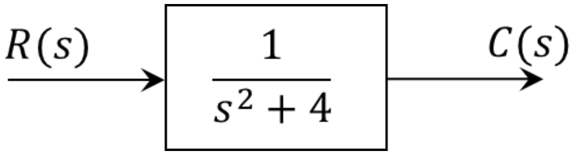
2. Frequency$= \dfrac{1}{T} = \dfrac{1}{\pi}\ Hz$

or 2 rad/s (based on ω_n; divide by 2pi to convert to Hz)

Question 2

Alternative method:

1. $R(s) = \dfrac{1}{s} \quad \Rightarrow \quad C(s) = \dfrac{1}{s^2+4} \cdot \dfrac{1}{s}$

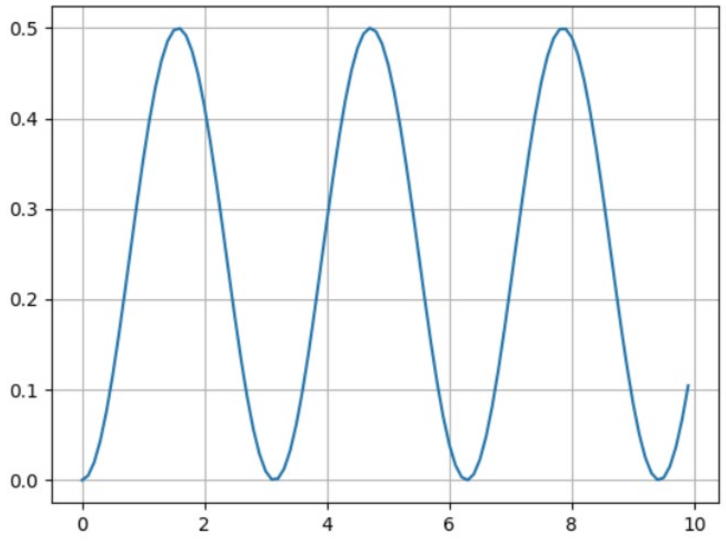


Using partial fractions: $\dfrac{1}{s(s^2+4)} = \dfrac{A}{s} + \dfrac{Bs+C}{(s^2+4)}$

$$\Rightarrow 1 = A(s^2+4) + (Bs+C)(s)$$

Compare coefficients,

$$\left.\begin{aligned} s^2&: \quad A + B = 0 \\ s^1&: \quad C = 0 \\ s^0&: \quad 4A = 1 \end{aligned}\right\} \quad A = \frac{1}{4} \quad B = -\frac{1}{4} \quad C = 0$$



$$C(s) = \frac{1}{4s} - \frac{1}{4}\frac{s}{(s^2+4)}$$

$$c(t) = \frac{1}{4} - \frac{1}{4}\cos 2t$$

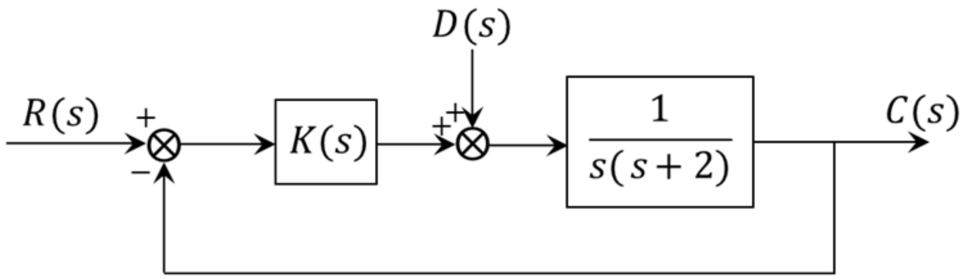
Table of Laplace Transforms

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}$	2.	e^{at}	$\dfrac{1}{s-a}$
3.	$t^n, \quad n=1,2,3,\ldots$	$\dfrac{n!}{s^{n+1}}$	4.	$t^p, p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\dfrac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}}, \quad n=1,2,3,\ldots$	$\dfrac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\dfrac{a}{s^2+a^2}$	8.	$\cos(at)$	$\dfrac{s}{s^2+a^2}$

Question 3

Consider the following system with controller $K(s)$, input $R(s)$ and disturbance $D(s)$.

1. Suppose that $K(s)$ is a proportional controller, i.e. $K(s) = K$. For which value of K the response to unit-ramp input has a steady-state error of 0.1?
2. If we want the response to unit-ramp input to have zero steady-state error, what form should the controller $K(s)$ take?
3. If we want the response to unit-step disturbance to be zero at steady-state, what form should the controller $K(s)$ take?



1. $D(s) = 0 \Rightarrow$ Open loop transfer function: $G(s) = \dfrac{K}{s(s+2)}$

(Unity-feedback system)

System is type 1 $\Rightarrow$ For a unit ramp input, $e_{SS} = 1/K_v$.

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{sK}{s(s+2)} = \frac{K}{2}$$

For $e_{SS} = 0.1$ to unit ramp input:

$$e_{SS} = \frac{1}{K_v} = \frac{2}{K} = 0.1 \quad \Rightarrow \quad K = 20$$

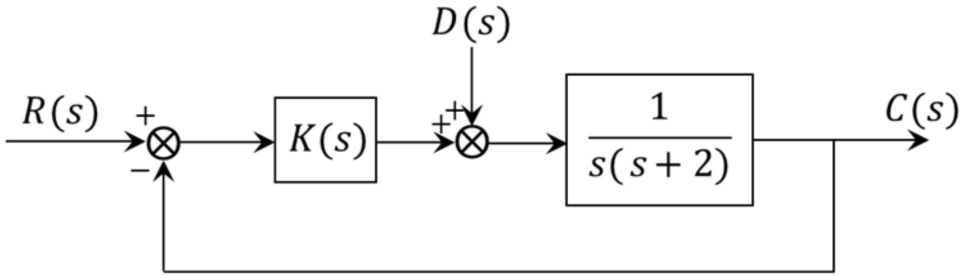
2.

To get $e_{SS} = 0$ for a unit ramp input, we must have at least a type 2 system.

Since there is already one integrator in the plant, the controller $K(s)$ should include another integrator.

Question 3

3. If we want the response to unit-step disturbance to be zero at steady-state, what form should the controller $K(s)$ take?



Assuming $R(s) = 0$:

$$\frac{C(s)}{D(s)} = \frac{\frac{1}{s(s+2)}}{1 + \frac{K(s)}{s(s+2)}} = \frac{\frac{1}{s(s+2)}}{\frac{s^2 + 2s + K(s)}{s(s+2)}} = \frac{1}{s^2 + 2s + K(s)}$$

Steady state output in response to a unit-step disturbance input, $D(s) = \frac{1}{s}$ is:

$$c_{SS} = \lim_{s\to 0} sC(s) = \lim_{s\to 0} \frac{sD(s)}{s^2 + 2s + K(s)} = \lim_{s\to 0} \frac{s \cdot \frac{1}{s}}{s^2 + 2s + K(s)} = \lim_{s\to 0} \frac{1}{K(s)}$$

Therefore, if $K(s)$ includes an integrator ($1/s$ term), then the output will have zero steady state response to a step disturbance, which is what we want.