NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

MA3005 Control Theory

Tutorial 5

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Question 1

Consider the closed-loop system given by $\dfrac{C(s)}{R(s)} = \dfrac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$

Determine the values of ζ and ω_n so that the system responds to a step input with approximately 5% overshoot and with a settling time of 2 sec. (Problem B-4-6 of text.)

Given that $M_p = 0.05$, and $t_s = 2$ seconds:

(Using two percent criterion)

$$t_s = 4T = \frac{4}{\zeta\omega_n} = 2 \quad \Rightarrow \quad \zeta\omega_n = 2$$

$T = \frac{1}{\zeta\omega_n}$ is the time constant of an underdamped 2nd order system

ln both sides

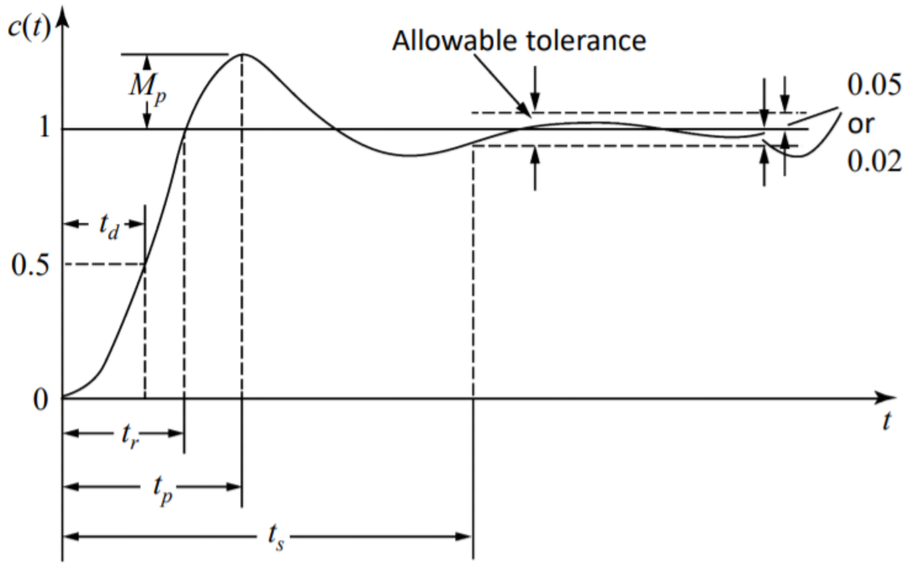
$$M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = 0.05 \quad \Rightarrow \quad \frac{-\zeta\pi}{\sqrt{1-\zeta^2}} = \ln 0.05 = -2.9957$$

$$\zeta^2 = \frac{2.9957^2}{\pi^2}(1-\zeta^2)$$

$$\zeta^2 = 0.9093\,(1-\zeta^2)$$

$$\zeta = 0.69$$

$$\Rightarrow \omega_n = \frac{2}{\zeta} = 2.9 \text{ rad/s}$$



Unit-step response of an underdamped second-order system

Question 2

Referring to the system shown below, determine the values of K and k such that the system has a damping ratio ζ of 0.7 and an undamped natural frequency ω_n of 4 rad/sec. (Problem B-4-12 of text.)

CLTF of system: $$\frac{C(s)}{R(s)} = \frac{K}{s^2 + (2 + Kk)s + K}$$

$$= \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

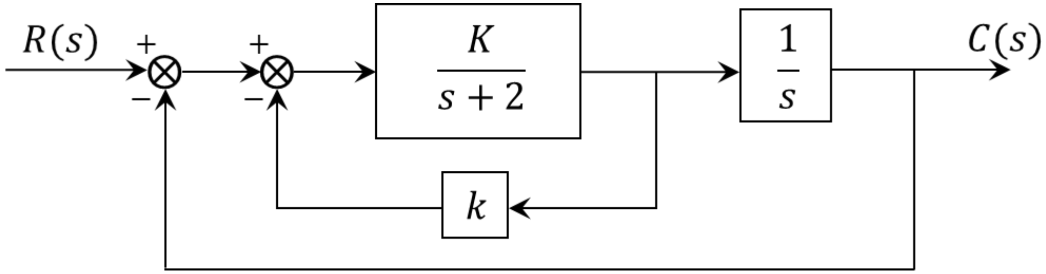
Comparing with the 2nd order function:

$$\Rightarrow \quad K = {\omega_n}^2 = 16$$

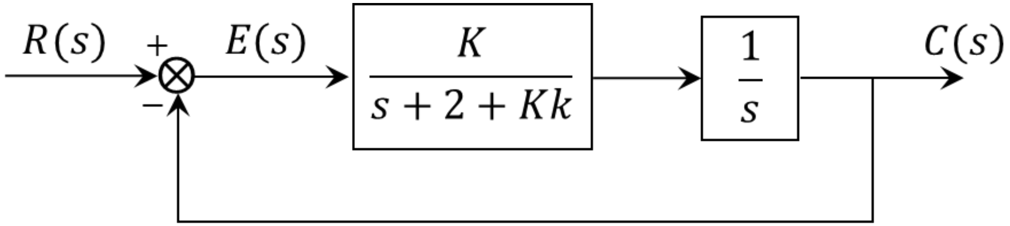
$$\Rightarrow \quad 2 + Kk = 2\zeta\omega_n$$

$$2 + (16)k = 2(0.7)(4)$$

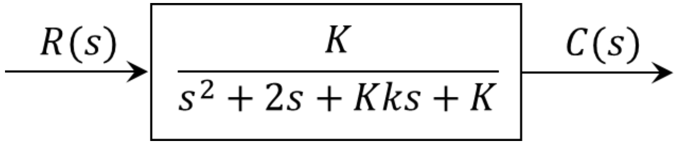
$$k = 0.225$$



$$TF = \frac{\frac{K}{s+2}}{1 + \frac{Kk}{s+2}} = \frac{\frac{K}{s+2}}{\frac{s+2+Kk}{s+2}} = \frac{K}{s+2+Kk}$$



$$TF = \frac{\frac{K}{s(s+2+Kk)}}{1 + \frac{K}{s(s+2+Kk)}} = \frac{\frac{K}{s(s+2+Kk)}}{\frac{s(s+2+Kk)+K}{s(s+2+Kk)}} = \frac{K}{s^2+2s+Kks+K}$$



Question 3

Consider the system shown in Figure (a) below. From the system, we can evaluate that the damping ratio of this system is 0.158 and the undamped natural frequency is 3.16 rad/sec (check it!).

To improve the relative stability, we employ tachometer feedback. Figure (b) shows such a tachometer-feedback system.

Determine the value of K_h so that the damping ratio of the system is 0.5. Draw unit-step response curves of both the original and tachometer-feedback systems. Also draw the error-versus-time curves for the unit-ramp response of both systems. (Problem B-4-3 of text.)

Check:

$$\frac{C(s)}{R(s)} = \frac{\frac{10}{s(s+1)}}{1+\frac{10}{s(s+1)}} = \frac{\frac{10}{s(s+1)}}{\frac{s^2+s+10}{s(s+1)}} = \frac{10}{s^2+s+10}$$

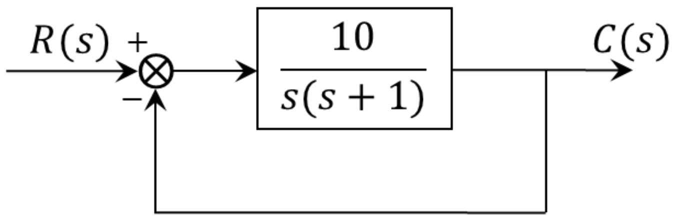
$$= \frac{{\omega_n}^2}{s^2+2\zeta\omega_n s+{\omega_n}^2}$$

$$\Rightarrow \quad {\omega_n}^2 = 10$$

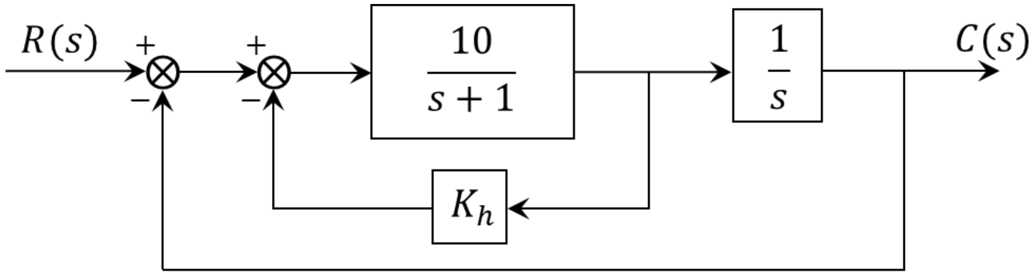
$$\omega_n = \sqrt{10} = 3.16 \text{ rad/s}$$

$$\Rightarrow \quad 2\zeta\omega_n = 1$$

$$\zeta = \frac{1}{2\zeta\omega_n} = \frac{1}{2(3.16)} = 0.158$$



(a)



(b)

Question 3

Determine K_h so that ζ is 0.5

CLTF of system: $$\frac{C(s)}{R(s)} = \frac{10}{s^2 + (1 + 10K_h)s + 10}$$

$$= \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

$$\Rightarrow \quad {\omega_n}^2 = 10$$

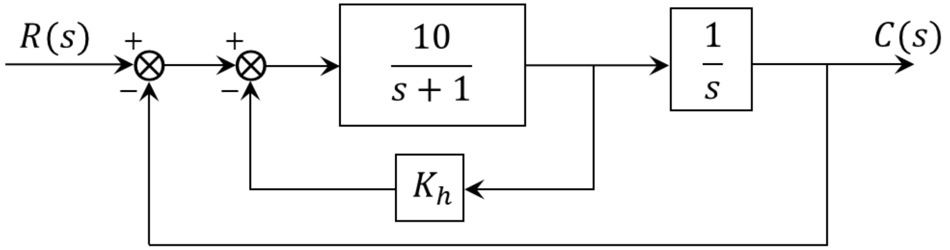
$$\omega_n = \sqrt{10} = 3.16 \text{ rad/s}$$

$$\Rightarrow \quad 2\zeta\omega_n = 1 + 10K_h$$

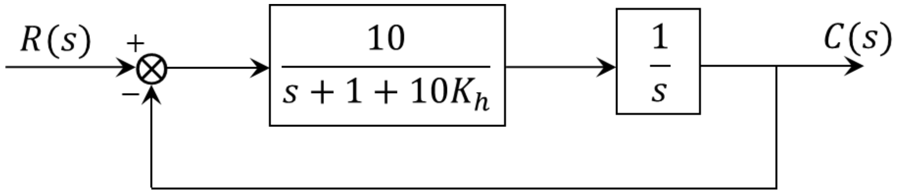
$$K_h = \frac{2\zeta\omega_n - 1}{10} = \frac{2(0.5)(3.16) - 1}{10} = 0.216$$

$$\Rightarrow \quad \frac{C(s)}{R(s)} = \frac{10}{s^2 + 3.16s + 10}$$

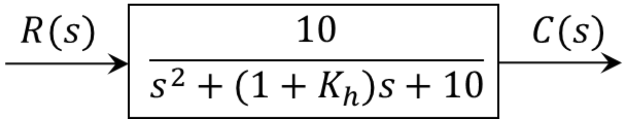
(b)



$$TF = \frac{\frac{10}{s+1}}{1 + \frac{10K_h}{s+1}} = \frac{\frac{10}{s+1}}{\frac{s+1+10K_h}{s+1}} = \frac{10}{s+1+10K_h}$$

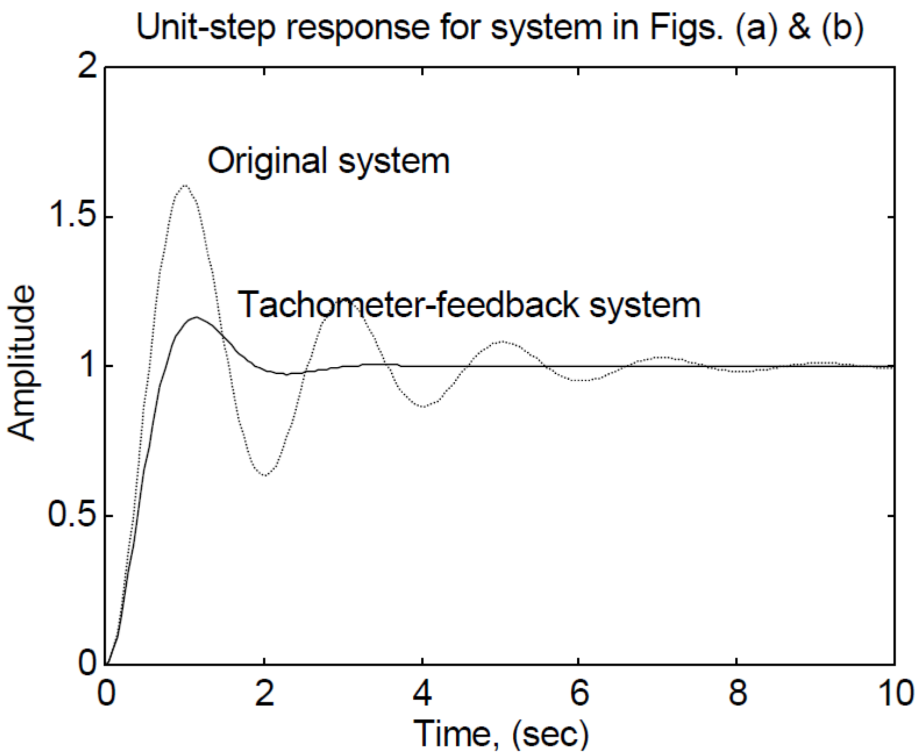


$$TF = \frac{\frac{10}{s(s+1+10K_h)}}{1 + \frac{10}{s(s+1+10K_h)}} = \frac{\frac{10}{s(s+1+10K_h)}}{\frac{s(s+1+10K_h)+10}{s(s+1+10K_h)}} = \frac{10}{s^2 + (1+10K_h)s + 10}$$

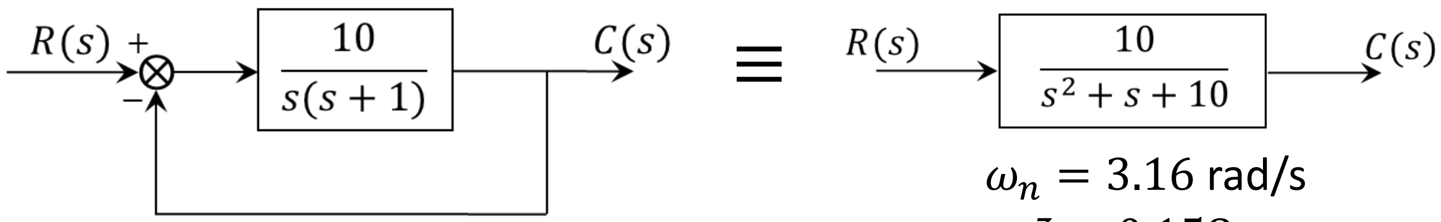


Question 3

Draw unit-step response curves of both the original and tachometer-feedback systems.



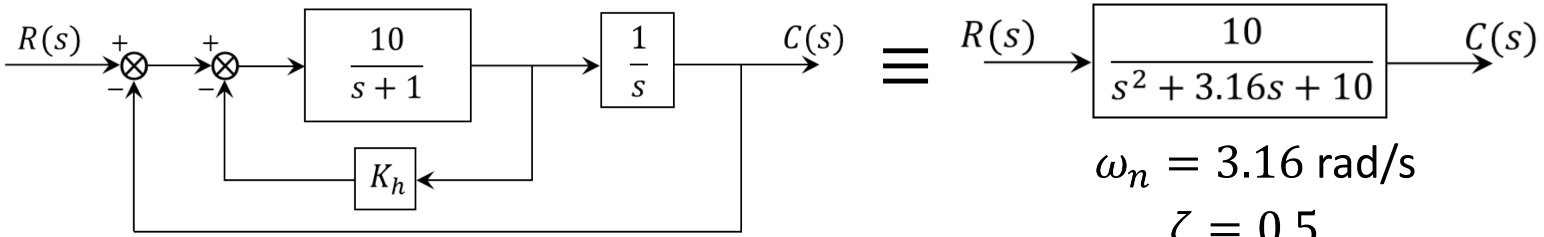
(a) Original system



$\omega_n = 3.16$ rad/s

$\zeta = 0.158$

(b) Tachometer-feedback system



$\omega_n = 3.16$ rad/s

$\zeta = 0.5$

With improved damping ratio (ζ), both the max overshoot (M_p) and time to reach steady state (t_s) have improved.

Note: Tachometer-feedback is one of the methods used to improve ζ without affecting undamped natural frequency (ω_n).

MATLAB script for plotting unit-step response curves:

```
num=[10];
den1=[1 3.16 10];
sys1=tf(num,den1);
den2=[1 1 10];
sys2=tf(num,den2);

time=0:0.1:10;
y1=step(sys1,time);
y2=step(sys2,time);
figure(1)
plot(time, y1, time, y2);
```

Question 3

Also draw the error-versus-time curves for the unit-ramp response of both systems.

$$\frac{E_1(s)}{R(s)} = \frac{R(s)-C_1(s)}{R(s)} = 1 - \frac{C_1(s)}{R(s)} = 1 - \frac{10}{s^2+s+10} = \frac{s^2+s}{s^2+s+10}$$

Unit-ramp input:

$$E_1(s) = \frac{s^2+s}{s^2+s+10}\cdot R(s) = \frac{s^2+s}{s^2+s+10}\left(\frac{1}{s^2}\right) = \boxed{\frac{s^2+s}{s^3+s^2+10s}}\cdot\frac{1}{s}$$

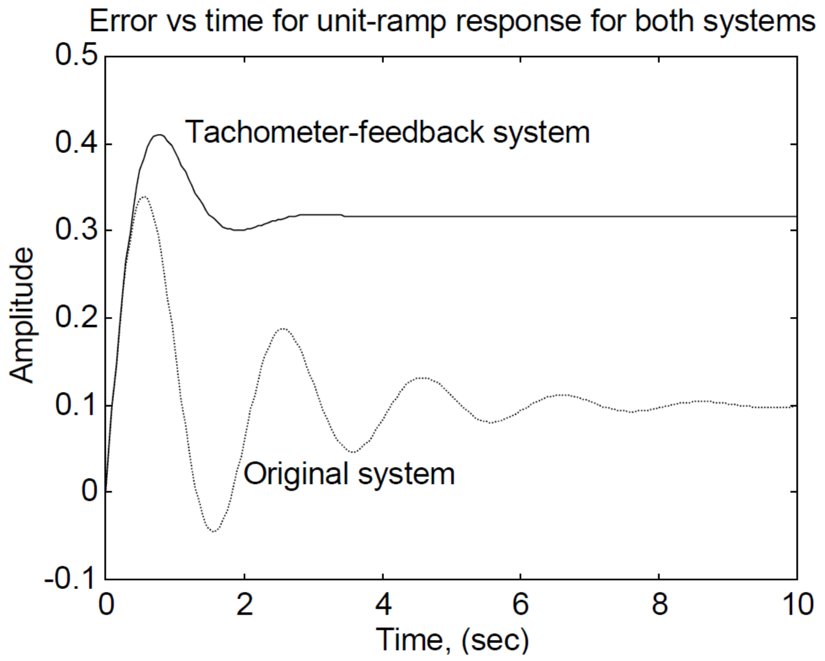
$$\frac{E_2(s)}{R(s)} = \frac{R(s)-C_2(s)}{R(s)} = 1 - \frac{C_2(s)}{R(s)} = 1 - \frac{10}{s^2+3.16s+10}$$

$$= \frac{s^2+3.16s}{s^2+3.16s+10}$$

Unit-ramp input:

$$E_2(s) = \frac{s^2+3.16s}{s^2+3.16s+10}\left(\frac{1}{s^2}\right) = \boxed{\frac{s^2+3.16s}{s^3+3.16s^2+10s}}\cdot\frac{1}{s}$$

The error-vs-time curves for the unit-ramp input for both systems will be the same as the unit step response for blue-boxed TF.



MATLAB script:

```
num1=[1 1 0];
den1=[1 1 10 0];
sys1=tf(num1,den1);
num2=[1 3.16 0];
den2=[1 3.16 10 0];
sys2=tf(num2,den2);

time=0:0.1:10;
y1=step(sys1,time);
y2=step(sys2,time);
figure(1)
plot(time, y1, time, y2);
```

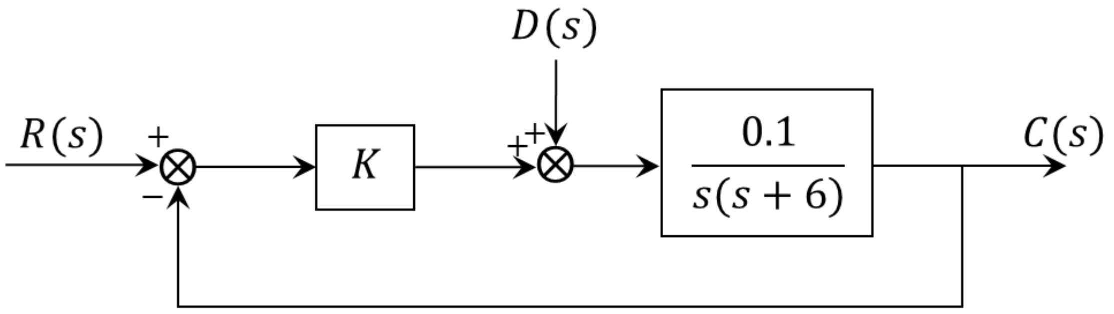
Question 4

In the figure shown below, $R(s)$ is an input that the output $C(s)$ should follow as closely as possible, and $D(s)$ is an unwanted disturbance to which (ideally) $C(s)$ should not respond at all. Find the transfer functions relating:

(a) $R(s)$ to $C(s)$ (assume that $D(s) = 0$)

(b) $D(s)$ to $C(s)$ (assume $R(s) = 0$)

Use the results of (a) and (b) to find the closed-loop steady state response to unit step inputs (c) the input $R(s)$, and (d) the disturbance $D(s)$. What happens to the two steady state responses as the controller gain K becomes very large? Find the value of the steady state response when $K = 80$. Does closed-loop operation reduce the effect of the disturbance?



(a)

$$\frac{C(s)}{R(s)} = \frac{\frac{0.1K}{s(s+6)}}{1+\frac{0.1K}{s(s+6)}} = \frac{\frac{0.1K}{s(s+6)}}{\frac{s^2+6s+0.1K}{s(s+6)}} = \frac{0.1K}{s^2+6s+0.1K}$$

(b)

$$\frac{C(s)}{D(s)} = \frac{\frac{0.1}{s(s+6)}}{1+\frac{0.1K}{s(s+6)}} = \frac{\frac{0.1}{s(s+6)}}{\frac{s^2+6s+0.1K}{s(s+6)}} = \frac{0.1}{s^2+6s+0.1K}$$

Question 4

Find the closed-loop steady state response to unit step inputs (c) the input $R(s)$, and (d) the disturbance $D(s)$. What happens to the two steady state responses as the controller gain K becomes very large? Find the value of the steady state response when $K = 80$. Does closed-loop operation reduce the effect of the disturbance?

The SS unit step response from $R(s)$ to $C(s)$ and $D(s)$ to $C(s)$ can be calculated from the Final value theorem:

(c) $$R(s) = \frac{1}{s} \quad \Rightarrow c_{R_{SS}} = \lim_{s\to 0} sC(s) = \lim_{s\to 0} s\frac{C(s)}{R(s)}R(s) = \lim_{s\to 0} s\frac{0.1K}{s^2 + 6s + 0.1K} \cdot \frac{1}{s} = 1$$

(d) $$D(s) = \frac{1}{s} \quad \Rightarrow c_{D_{SS}} = \lim_{s\to 0} sC(s) = \lim_{s\to 0} s\frac{C(s)}{D(s)}D(s) = \lim_{s\to 0} s\frac{0.1}{s^2 + 6s + 0.1K} \cdot \frac{1}{s} = \frac{1}{K}$$

K has no effect on the SS response (independent of K) to unit step input $R(s)$. However, the SS response to unit step input $D(s)$ is inversely proportional to K. As K becomes very large, its SS response becomes very small.

$$D(s) = \frac{1}{s} \quad \Rightarrow \quad c_{D_{SS}} = \lim_{s\to 0} sC(s) = \frac{1}{K} = \frac{1}{80} = 0.0125$$

The closed-loop operation can reduce the effect of the disturbance while maintaining good tracking property.