NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

MA3005 Control Theory

Tutorial 4

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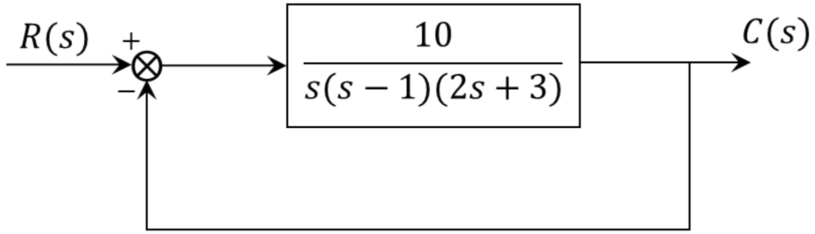
Question 1

Consider a unity-feedback control system with the following open-loop transfer function:

$$G(s) = \frac{10}{s(s-1)(2s+3)}$$



Is this system stable?

CLTF of system: $$\frac{C(s)}{R(s)} = \frac{\dfrac{10}{s(s-1)(2s+3)}}{1+\dfrac{10}{s(s-1)(2s+3)}} = \frac{10}{2s^3+s^2-3s+10}$$

Not easy to factorise higher order polynomials
⇒ Use Routh-Hurwitz method

CE: $2s^3 + s^2 - 3s + 10 = 0$

Routh's Array:

$$\begin{array}{c|c|c} s^3 & 2 & -3 \\ s^2 & 1 & 10 \\ s & b_1 = -23 & \\ s^0 & c_1 = 10 & \end{array}$$

$$b_1 = \frac{(1)(-3)-(2)(10)}{1} = -23$$

$$c_1 = \frac{(-23)(10)-(1)(0)}{-23} = 10$$

First column:

- If there is no changing of sign ⇒ stable.
- No. of sign changes ⇒ no. of poles in RHP

CE: $2s^3 + s^2 - 3s + 10 = 0$

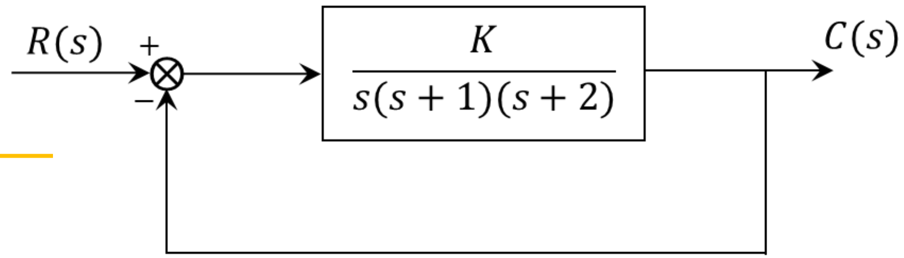
Check by solving: $s_1 = -2.21$
$s_{2,3} = 0.85 \pm j1.24$

Since there is 2 sign changes in the first column, the system is **unstable**.

Question 2

Determine the range of K for stability of a unit-feedback control system whose open-loop transfer function is

$$G(s) = \frac{K}{s(s+1)(s+2)}$$



CLTF of system: $$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}} = \frac{K}{s^3+3s^2+2s+K}$$

CE: $s^3 + 3s^2 + 2s + K = 0$

Routh's Array:

s^3	1	2
s^2	3	K
s	$b_1 = \frac{6-K}{3}$	
s^0	$c_1 = K$	

$$b_1 = \frac{(3)(2)-(1)(K)}{3} = \frac{6-K}{3}$$

$$c_1 = \frac{\left(\frac{6-K}{3}\right)(K)-(3)(0)}{\frac{6-K}{3}} = K$$

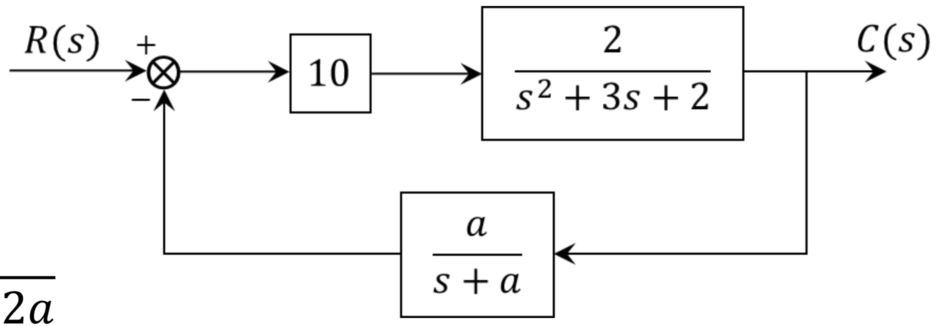
For stability, the coefficients in the first column should have the same sign:

$$6-K > 0 \quad \& \quad K > 0$$

$$\Rightarrow 6 > K > 0$$

Question 3

The block diagram shown below is a control system with a noise filter in its feedback loop. When a sensor signal is noisy, we often use a low-pass filter to smooth out the signal. Care must be taken, however, since such a filter may cause instability. Using the Routh-Hurwitz stability criterion, determine the range of the filter parameter a to keep the closed loop system stable.



CLTF of system:

$$\frac{C(s)}{R(s)} = \frac{\frac{20}{s^2+3s+2}}{1+\left(\frac{a}{s+a}\right)\left(\frac{20}{s^2+3s+2}\right)} = \frac{20(s+a)}{s^3+(3+a)s^2+(2+3a)s+22a}$$

CE: $s^3+(3+a)s^2+(2+3a)s+22a=0$

Routh's Array:

s^3	1	$2+3a$
s^2	$3+a$	$22a$
s	$\frac{3a^2-11a+6}{3+a}$	
s^0	$22a$	

$$b_1 = \frac{(3+a)(2+3a) - (1)(22a)}{3+a} = \frac{3a^2-11a+6}{3+a}$$

$$c_1 = \frac{\left(\frac{3a^2-11a+6}{3+a}\right)(22a) - (3+a)(0)}{\frac{3a^2-11a+6}{3+a}} = 22a$$

Question 3

Continue.. determine the range of the filter parameter a to keep the closed loop system stable.

CE: $s^3 + (3+a)s^2 + (2+3a)s + 22a = 0$ $\Rightarrow$ necessary condition: $a > 0$
(since coeff of s^3:1)

<u>Necessary (but not sufficient) condition</u>

If a system is stable, then all coefficients of its CE are real, are of the same sign, and none of its coefficients is zero.

Routh's Array:

s^3	1	$2+3a$
s^2	$3+a$	$22a$
s	$\dfrac{3a^2-11a+6}{3+a}$	
s^0	$22a$	

For stability, coefficients in 1st column should have <u>positive</u> sign:

1) $3+a>0 \quad\Rightarrow\quad a>-3 \quad\Rightarrow\quad \boxed{a>0}$

2) $\dfrac{3a^2-11a+6}{3+a} = \dfrac{(3a-2)(a-3)}{3+a} > 0 \quad\Rightarrow\quad (3a-2)(a-3)>0$
$\Rightarrow \boxed{a<\frac{2}{3} \quad \& \quad a>3}$

3) $22a>0 \quad\Rightarrow\quad \boxed{a>0}$

Together, $\boxed{0<a<\frac{2}{3} \quad \& \quad a>3}$

Question 4

A feedback control system has a characteristic equation: $s^3 + (1+K)s^2 + 10s + (5+15K) = 0$

The parameter K must be positive. What is the maximum value K can assume before the system becomes unstable ? When K is equal to the maximum value, the system oscillates. Determine the frequency of oscillation.

Routh's Array:

$$\begin{array}{c|c|c} s^3 & 1 & 10 \\ s^2 & 1+K & 5+15K \\ s & \dfrac{5-5K}{1+K} & \\ s^0 & 5+15K & \end{array}$$

$$b_1 = \frac{(1+K)(10) - (1)(5+15K)}{1+K} = \frac{5-5K}{1+K}$$

$$c_1 = \frac{\left(\dfrac{5-5K}{1+K}\right)(5+15K) - (1+K)(0)}{\dfrac{5-5K}{1+K}} = 5+15K$$

Coefficients in 1st column should have the same sign:

1) $1+K>0 \;\Rightarrow\; K>-1 \;\Rightarrow\; \boxed{K>0}$

2) $\dfrac{5-5K}{1+K}>0 \;\Rightarrow\; 5-5K>0 \;\Rightarrow\; \boxed{K<1}$

3) $5+15K>0 \;\Rightarrow\; K>-\frac{1}{3} \;\Rightarrow\; \boxed{K>0}$

$\boxed{0<K<1}$

Max $K = 1$

If $K = 1$, entire 3rd row becomes zero

$\Rightarrow$ Forms an auxiliary polynomial:

$$(1+K)s^2 + (5+15K) = 0$$

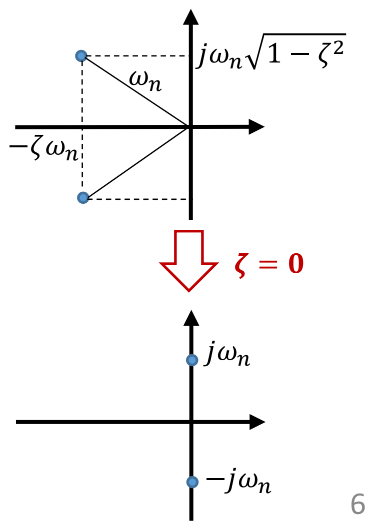
$$2s^2 + 20 = 0$$

$$\Rightarrow\; s = \pm j\sqrt{10}$$

∴ Frequency of oscillation is $\sqrt{10}$ rad/s.

Second-order systems

$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$



Bonus Questions

Use Routh's stability criterion to determine how many roots with positive real parts the following equations have.

a) $s^4 + 8s^3 + 32s^2 + 80s + 100 = 0$

$$
\begin{array}{llll}
s^4 & 1 & 32 & 100 \\
s^3 & 8 & 80 & \\
s^2 & \frac{8(32)-1(80)}{8} = 22 & \frac{8(100)-1(0)}{8} = 100 & \\
s^1 & \frac{22(80)-8(100)}{22} = 43.6 & & \\
s^0 & \frac{43.6(100)-22(0)}{43.6} = 100 & &
\end{array}
$$

Since there is no sign change in the first column, no RHP pole exists and the system is stable.

b) $s^5 + 10s^4 + 30s^3 + 80s^2 + 344s + 480 = 0$

$$
\begin{array}{llll}
s^5 & 1 & 30 & 344 \\
s^4 & 10 & 80 & 480 \\
s^3 & \frac{10(30)-1(80)}{10} = 22 & \frac{10(344)-1(480)}{10} = 296 & \\
s^2 & \frac{22(80)-10(296)}{22} = -54.5 & \frac{22(480)-10(0)}{22} = 480 & \\
s^1 & \frac{-54.5(296)-22(480)}{-54.5} = 489.6 & & \\
s^0 & 480 & &
\end{array}
$$

Since there are 2 sign changes in the first column, 2 RHP poles exist and the system is unstable.

Bonus Questions

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c) $s^4 + 2s^3 + 7s^2 - 2s + 8 = 0$

s^4	1	7	8
s^3	2	-2	
s^2	$\frac{2(7)-1(-2)}{2} = 8$	$\frac{2(8)-1(0)}{2} = 8$	
s^1	$\frac{8(-2)-2(8)}{8} = -4$		
s^0	8		

Since there are 2 sign changes in the first column, 2 RHP poles exist and the system is unstable.

d) $s^4 + 6s^2 + 25 = 0$

s^4	1	6	25
s^3	$\cancel{0}$ 4	$\cancel{0}$ 12	
s^2	$\frac{4(6)-1(12)}{4} = 3$	$\frac{4(25)-1(0)}{4} = 25$	
s^1	$\frac{3(12)-4(25)}{3} = -21.3$		
s^0	25		

If the entire row of array is zero, there exist pairs of roots, which are mirror images w.r.t the imaginary axis. The system is never stable (either unstable or marginally stable)

- Form auxiliary polynomial $P(s)$ (based on the row above the zero row array) and create new row by using coeff of $\frac{dP(s)}{ds}$

  $P(s) = s^4 + 6s^2 + 25 \Rightarrow \frac{dP(s)}{ds} = 4s^3 + 12s$

Since there are 2 sign changes in the first column, 2 RHP poles exist and the system is unstable.

Bonus Questions

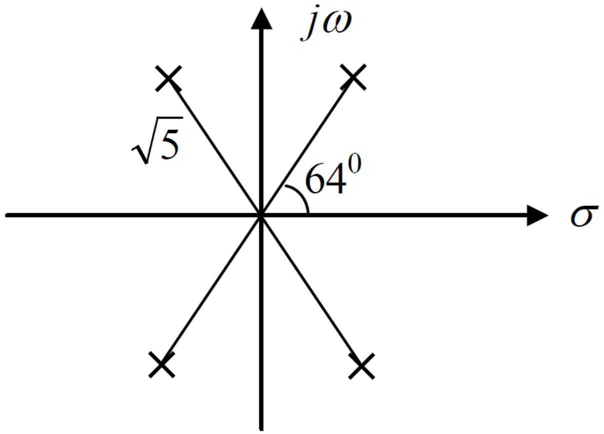
$s^4 + 6s^2 + 25 = 0$

Unstable system

To verify the answer by checking the solution to the auxiliary polynomial, i.e let $P(s) = 0$ and solve for s.

$$P(s) = s^4 + 6s^2 + 25 = 0 \quad \Rightarrow \quad s^2 = \frac{-6 \pm \sqrt{36 - 100}}{2} = -3 \pm j4 = 5e^{j(\pi \mp 0.93) + 2n\pi j}$$

$$s = \sqrt{5}e^{j\left(\frac{\pi}{2} \mp 0.46\right) + n\pi j} \quad \text{where } n = 0,1$$



Two pairs of complex conjugate poles which are mirror image of the $j\omega$ axis.