NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

MA3005 Control Theory

Tutorial 3

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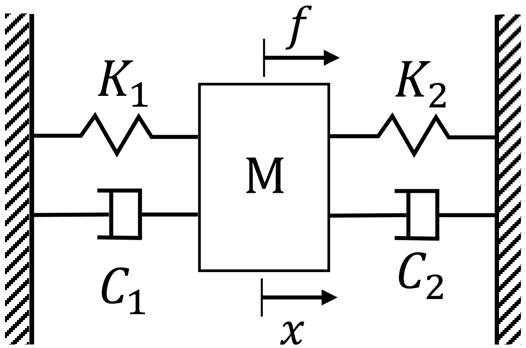
Question 1

Write the differential equations for the mechanical system, where f is force input and x is displacement output. Find the transfer function for input to output.

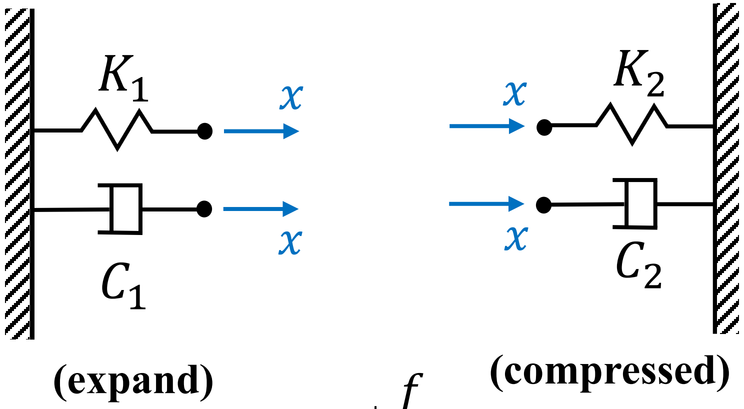


Newton's second law:

$$\sum F = m\ddot{x}$$

$$f - K_1 x - K_2 x - C_1\dot{x} - C_2\dot{x} = m\ddot{x}$$

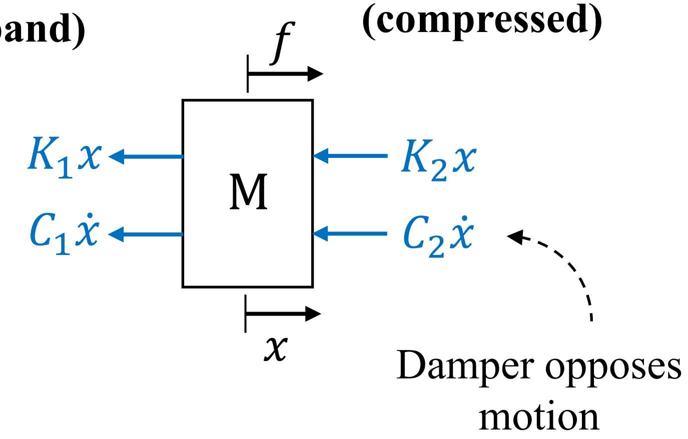
$$m\ddot{x} + (C_1 + C_2)\dot{x} + (K_1 + K_2)x = f$$



Taking Laplace transform and assuming zero initial condition:

$$\boxed{\begin{array}{c}\mathcal{L}\{\ddot{x}(t)\} = s^2X(s) - sx(0) - \dot{x}(0) \\ \mathcal{L}\{\dot{x}(t)\} = sX(s) - x(0)\end{array}}$$

$$m\left(s^2X(s) - \cancel{sx(0)} - \cancel{\dot{x}(0)}\right) + (C_1 + C_2)\left(sX(s) - \cancel{x(0)}\right) + (K_1+K_2)X(s) = F(s)$$

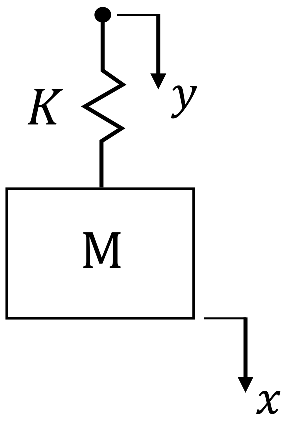


$$[ms^2 + (C_1 + C_2)s + (K_1 + K_2)]\, X(s) = F(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + (C_1 + C_2)s + (K_1 + K_2)}$$

Question 2

Write the differential equations for the mechanical system, if a displacement input y is given, which results in a displacement output is x. Find the transfer function.



Newton's second law:

$$\sum F = m\ddot{x}$$

$$-K(x-y) = m\ddot{x}$$

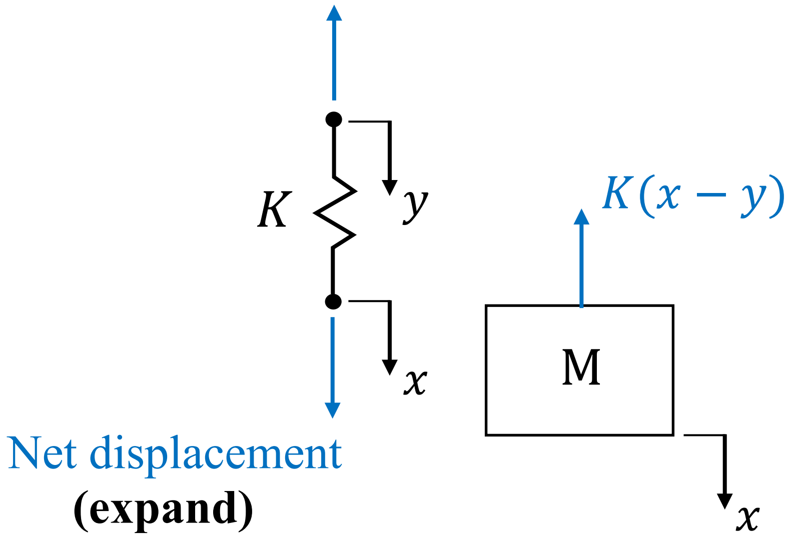
$$m\ddot{x} + Kx = Ky$$

Laplace transform (assuming zero initial condition):

$$[ms^2 + K]\,X(s) = K\,Y(s)$$

$$\frac{X(s)}{Y(s)} = \frac{K}{ms^2 + K}$$

Assume $x > y$



Background for Rotational System

Newton's second law:

$$\sum T = J\ddot{\theta}$$

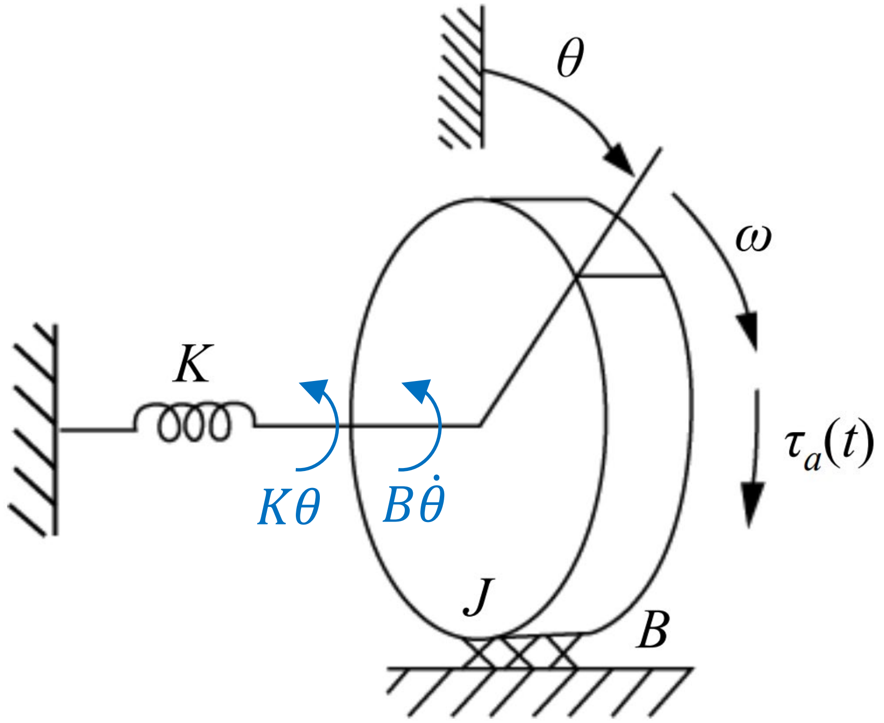
$$\tau_a - B\dot{\theta} - K\theta = J\ddot{\theta}$$

$$J\ddot{\theta} + B\dot{\theta} + K\theta = \tau_a$$

Taking Laplace transform and zero initial condition:

$$Js^2\Theta(s) + Bs\,\Theta(s) + K\Theta(s) = T(s)$$

$$\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K}$$



Note:

Torque τ_a is the input.

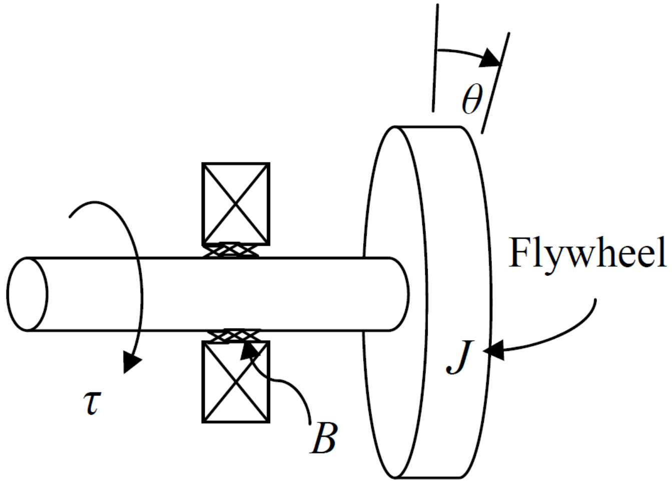
Angular displacement θ is the output.

Question 3

Consider the rotational system having a flywheel with inertia $J = 100\ kgm^2$ and viscous damping coefficient $B = 10\ Ns/rad$ shown in the figure.

a) Find the transfer function $\frac{\Omega(s)}{T(s)}$ for input torque τ to angular velocity ω

b) Using Laplace transform, find out the $\omega(t)$ if a torque of $100\ Nm$ is applied. What is the steady state value for $\omega(t)$?



a) Newton's second law:

$$J\ddot{\theta} + B\dot{\theta} + \cancel{K\theta} = \tau$$

$$J\ddot{\theta} + B\dot{\theta} = \tau$$

$$J\dot{\omega} + B\omega = \tau$$

Laplace transform:

$$Js\Omega(s) + B\Omega(s) = T(s)$$

$$\frac{\Omega(s)}{T(s)} = \frac{1}{Js + B} = \frac{1}{100s + 10}$$

b) If a torque of $100\ Nm$ is applied, then $T(s) = \frac{100}{s}$.

$$\Omega(s) = \frac{1}{100s + 10}T(s) = \frac{1}{100s + 10}\frac{100}{s}$$

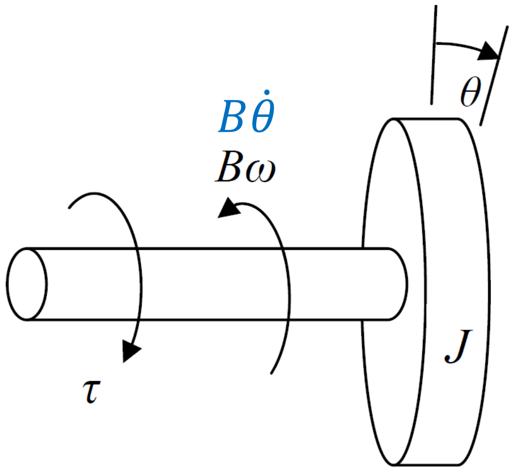
$$= \frac{1}{100s + 10}\frac{100}{s}$$

$$= \frac{1}{(s + 0.1)s}$$

$$= \frac{10}{s} - \frac{10}{(s + 0.1)}$$

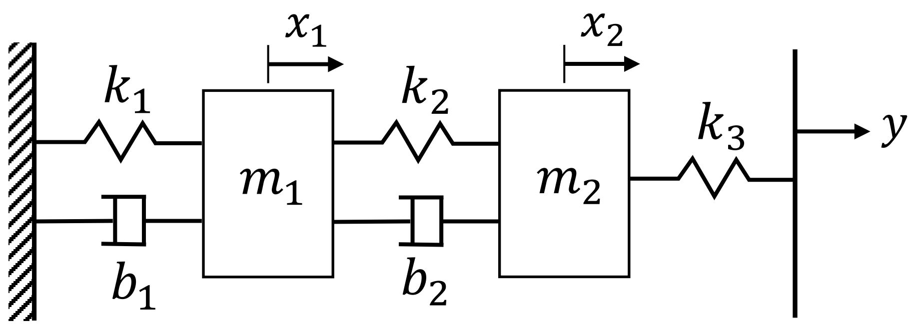
Inverse Laplace transform: $\omega(t) = 10 - 10e^{-0.1t}$

Steady state value: $t \to \infty, \omega(t) = 10$ (Can also be found using FVT)

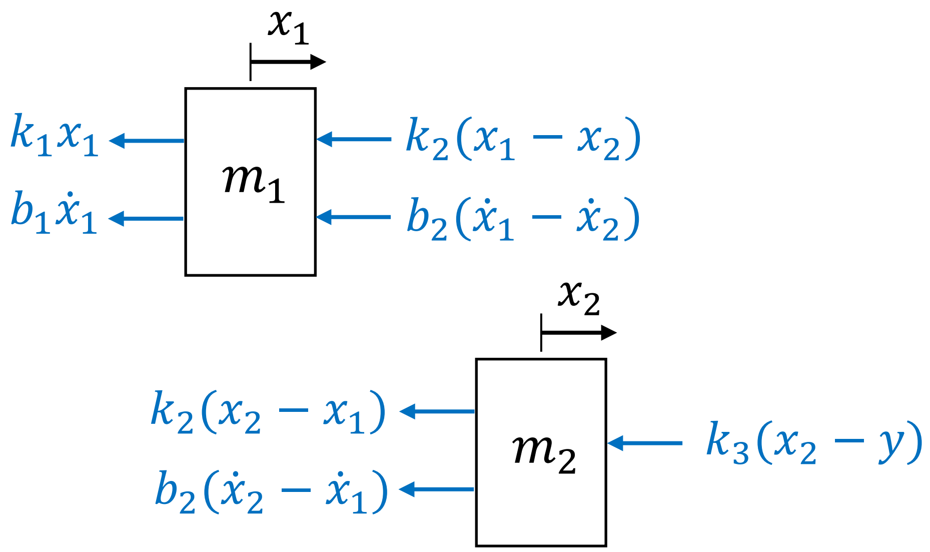


Bonus Question

Write the differential equations for the mechanical system.



For FBD, assume $x_1 > x_2$ for mass m_1 and $x_2 > x_1$ for mass m_2



Newton's second law for m_1 :

$$\sum F = m_1 \ddot{x}_1$$

$$-k_1 x_1 - k_2(x_1 - x_2) - b_1 \dot{x}_1 - b_2(\dot{x}_1 - \dot{x}_2) = m_1 \ddot{x}_1$$

$$m_1 \ddot{x}_1 + (b_1 + b_2)\dot{x}_1 + (k_1 + k_2)x_1 = b_2 \dot{x}_2 + k_2 x_2$$

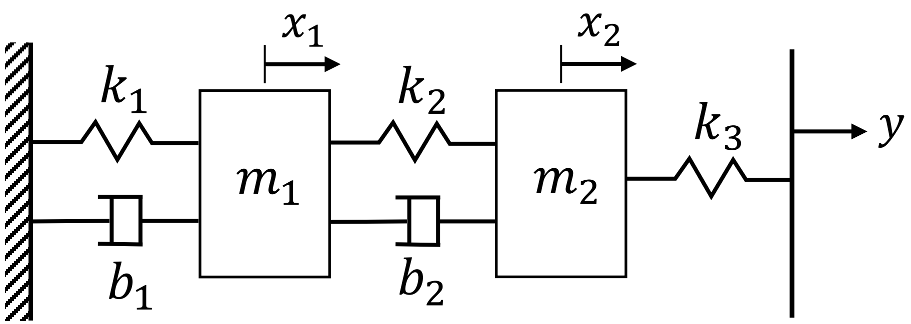
Newton's second law for m_2 :

$$-k_2(x_2 - x_1) - b_2(\dot{x}_2 - \dot{x}_1) - k_3(x_2 - y) = m_2 \ddot{x}_2$$

$$m_2 \ddot{x}_2 + b_2 \dot{x}_2 + (k_2 + k_3)x_2 = b_2 \dot{x}_1 + k_2 x_1 + k_3 y$$

Bonus Question

To find TF linking the output $X_1(s)$, $X_2(s)$ and input $Y(s)$:



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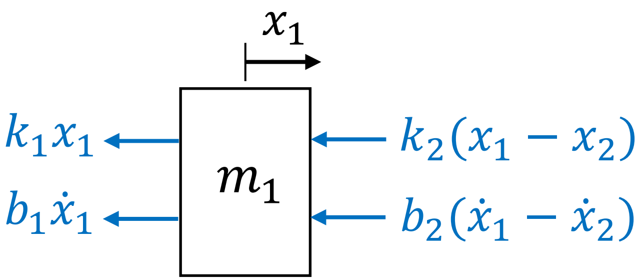

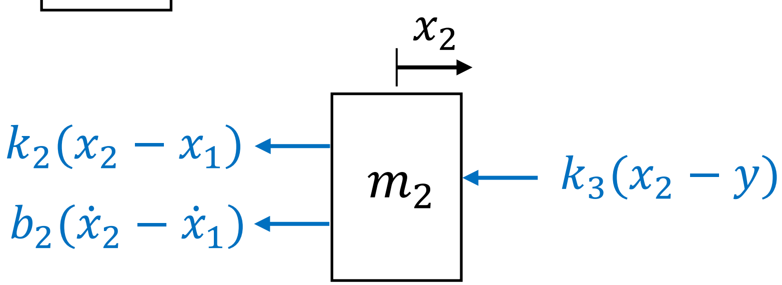

m_1 :

$$m_1\ddot{x}_1 + (b_1 + b_2)\dot{x}_1 + (k_1 + k_2)x_1 = b_2\dot{x}_2 + k_2 x_2$$

Laplace transform:

$$[m_1 s^2 + (b_1 + b_2)s + (k_1 + k_2)]X_1(s) = (b_2 s + k_2)X_2(s)$$

$$X_1(s) = \frac{(b_2 s + k_2)X_2(s)}{[m_1 s^2 + (b_1 + b_2)s + (k_1 + k_2)]}$$

m_2 :

$$m_2\ddot{x}_2 + b_2\dot{x}_2 + (k_2 + k_3)x_2 = b_2\dot{x}_1 + k_2 x_1 + k_3 y$$

Subst $X_1(s)$

Laplace transform:

$$[m_2 s^2 + b_2 s + (k_2 + k_3)]X_2(s) = (b_2 s + k_2)X_1(s) + k_3 Y(s)$$

$$\frac{X_2(s)}{Y(s)} = \cdots can\ be\ found$$

$$\frac{X_1(s)}{Y(s)} = \cdots can\ also\ be\ found$$