NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

MA3005 Control Theory

Tutorial Group MA3

Tutorial 1

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Question 1

By applying the final-value theorem, find the final value of $f(t)$ whose Laplace transform is given by

$$F(s) = \frac{10}{s(s+1)}$$

Verify this result by taking the inverse Laplace transform of $F(s)$ and letting $t \to \infty$. (Problem B-2-8 of text.)

Table of Laplace Transforms

	$f(t) = \mathfrak{L}^{-1}\{F(s)\}$	$F(s) = \mathfrak{L}\{f(t)\}$
1.	$1(t)$, unit step	$\frac{1}{s}$
2.	e^{at}	$\frac{1}{s-a}$

By final value theorem,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$= \lim_{s\to 0} \frac{10}{(s+1)} = 10$$

By partial fraction expansion, let

$$\frac{A}{s} + \frac{B}{s+1} = \frac{10}{s(s+1)}$$

$$\therefore \qquad 10 = A(s+1) + Bs = (A+B)s + A$$

$$\Rightarrow \qquad A = 10, \qquad B = -10$$

Therefore, $F(s) = \dfrac{10}{s} - \dfrac{10}{s+1}$

$$F(s) = \frac{10}{s} - \frac{10}{s+1}$$

taking the inverse Laplace transform

$$f(t) = 10 \bullet 1(t) - 10\, e^{-t}$$

letting $t \to \infty$, $\lim_{t\to\infty} f(t) = 10$

Question 2

Given

$$F(s) = \frac{1}{(s+2)^2}$$

determine the values of $f(0+)$ and $\dot{f}(0+)$. (Use the initial-value theorem.) (Problem B-2-9 of text.)

Using the initial value theorem,

$$f(0^+) = \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$$

$$= \lim_{s\to\infty} \frac{s}{(s+2)^2} = 0$$

Now, let $\dot{f}(t) = g(t)$

$$\mathcal{L}[g(t)] = G(s) = \mathcal{L}[\dot{f}(t)]$$

$$= sF(s) - f(0^+)$$

$$= \frac{s}{(s+2)^2}$$

$$\dot{f}(0^+) = g(0^+) = \lim_{t\to 0^+} g(t) = \lim_{s\to\infty} sG(s)$$

$$= \lim_{s\to\infty} s[sF(s) - f(0^+)]$$

$$= \lim_{s\to\infty} \frac{s^2}{s^2+4s+4} = 1$$

Question 3

Using partial-fraction expansions, find the function $f(t)$ of the following

(a) $$F(s) = \frac{10}{s(s+1)(s+10)}$$

$$\frac{\alpha}{s} + \frac{\beta}{s+1} + \frac{\gamma}{s+10} = \frac{10}{s(s+1)(s+10)}$$

$$\therefore \quad \alpha(s+1)(s+10) + \beta s(s+10) + \gamma s(s+1) = 10$$

$$s = 0, \Rightarrow \alpha = 1;$$

$$s = -1, \Rightarrow \beta = -\frac{10}{9};$$

$$s = -10, \Rightarrow \gamma = \frac{1}{9}$$

$$F(s) = \frac{1}{s} - \frac{\frac{10}{9}}{s+1} + \frac{\frac{1}{9}}{s+10} \quad \Rightarrow \quad \mathcal{L}^{-1}[F(s)] = f(t) = \left\{1 - \frac{10}{9}e^{-t} + \frac{1}{9}e^{-10t}\right\}1(t)$$

Table of Laplace Transforms

	$f(t) = \mathfrak{L}^{-1}\{F(s)\}$	$F(s) = \mathfrak{L}\{f(t)\}$
1.	$1(t)$, unit step	$\frac{1}{s}$
2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$

Question 3

Using partial-fraction expansions, find the function $f(t)$ of the following

(b) $$F(s) = \frac{1}{s(s+2)^2}$$

$$\frac{\alpha}{s} + \frac{\beta}{s+2} + \frac{\gamma}{(s+2)^2} = \frac{1}{s(s+2)^2}$$

$$\therefore \ \alpha(s+2)^2 + \beta s(s+2) + \gamma s = 1$$

$$s = 0, \quad \Rightarrow \quad \alpha = \frac{1}{4};$$

$$s = \text{-}2, \quad \Rightarrow \quad \gamma = -\frac{1}{2}$$

$$s = 1, \quad \Rightarrow \quad \beta = -\frac{1}{4}$$

$$F(s) = \frac{\frac{1}{4}}{s} - \frac{\frac{1}{4}}{s+2} - \frac{\frac{1}{2}}{(s+2)^2}$$

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{4}\left\{1 - e^{-2t} - 2te^{-2t}\right\}1(t)$$

Table of Laplace Transforms

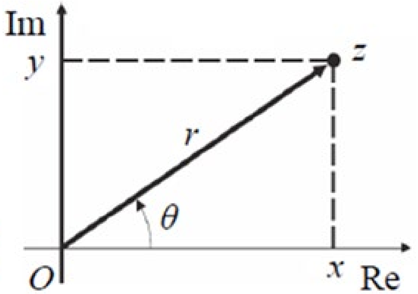
$f(t) = \mathfrak{L}^{-1}\{F(s)\}$	$F(s) = \mathfrak{L}\{f(t)\}$
$1(t)$, unit step	$\frac{1}{s}$
$\mathbf{e}^{at}$	$\frac{1}{s-a}$
$t^n\mathbf{e}^{at}, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{(s-a)^{n+1}}$

Background for Question 4

- A *complex number z*

  $\boxed{z = x + jy}$ $\quad j = \sqrt{-1}$

  - $x = \mathrm{Re}(z)$: real part of z
  - $y = \mathrm{Im}(z)$: imaginary part of z
  - $|z| = \sqrt{x^2 + y^2}$: *magnitude*
  - $\theta = \tan^{-1}\dfrac{y}{x}$: *phase angle* from Re(+) axis (positive ccw)



Complex function: function of complex variable

- Also has real and imaginary parts

  $$\boxed{G(s) = \mathrm{Re}(G(s)) + j\,\mathrm{Im}(G(s))}$$

- Magnitude and phase of complex function:

  $$|G(s)| = \sqrt{\mathrm{Re}(G(s))^2 + \mathrm{Im}(G(s))^2}$$

  $$\angle G(s) = \tan^{-1}\frac{\mathrm{Im}(G(s))}{\mathrm{Re}(G(s))}$$

- Special type of complex function in control problem:

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = \frac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$$

$$\boxed{|G(s)| = \frac{\prod \text{zero lengths}}{\prod \text{pole lengths}} = \frac{|K|\prod_{i=1}^{m}|s+z_i|}{\prod_{j=1}^{n}|s+p_j|}}$$

$$\boxed{\begin{aligned}\angle G(s) = \theta &= \sum \text{zero angles} - \sum \text{pole angles} \\ &= \sum_{i=1}^{m}\angle(s+z_i) - \sum_{j=1}^{n}\angle(s+p_j)\end{aligned}}$$

Question 4

For the following complex function

$$G(s) = \frac{s+2}{s^2(s+10)}$$

obtain the expression of the function in terms of

(a) real $\mathrm{Re}[G(j\omega)]$ and imaginary $\mathrm{Im}[G(j\omega)]$ parts;

(b) magnitude $|G(j\omega)|$ and angle $\angle G(j\omega)$

(a) Substituting for $s = j\omega$

$$G(j\omega) = \frac{j\omega+2}{(j\omega)^2(j\omega+10)}$$

$$= -\frac{(j\omega+2)}{\omega^2(j\omega+10)} \cdot \frac{(-j\omega+10)}{(-j\omega+10)}$$

$$= -\frac{(20+\omega^2+j8\omega)}{\omega^2(\omega^2+100)}$$

$$= -\frac{(20+\omega^2)}{\omega^2(\omega^2+100)} + j\frac{-8\omega}{\omega^2(\omega^2+100)}$$

(b)

$$G(j\omega) = \frac{j\omega+2}{(j\omega)^2(j\omega+10)}$$

$$|G(j\omega)| = \frac{|j\omega+2|}{|(j\omega)^2||(j\omega+10)|}$$

$$= \frac{\sqrt{(\omega^2+4)}}{\omega^2\sqrt{(\omega^2+100)}}$$

$$\angle(G(j\omega)) = \tan^{-1}\left(\frac{\omega}{2}\right) - 180^\circ - \tan^{-1}\left(\frac{\omega}{10}\right)$$