NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

# MA3005 Control Theory

## Tutorial 11

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# Question 1

Sketch the Bode plots (gain and phase shift) of the transfer function $G(s) = \dfrac{10(s+3)}{s(s^2+s+2)}$. Detail every step.

$$G(s) = \frac{10(s+3)}{s(s^2+s+2)} = 10\ (s+3)\left(\frac{1}{s}\right)\frac{1}{(s^2+s+2)} = 15\left(\frac{s+3}{3}\right)\left(\frac{1}{s}\right)\frac{2}{(s^2+s+2)}$$

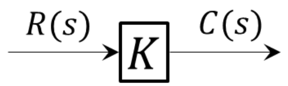
| Constant | First order zero | Integrator | Second order pole |
|---|---|---|---|
| $K$ | $\dfrac{s+a}{a}$ | $\dfrac{1}{s}$ | $\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ |

- Draw 'straight lines' magnitude and phase plots for each principal component
- Superposition of the plots

# Question 1

$$G(s) = 15\left(\frac{s+3}{3}\right)\left(\frac{1}{s}\right)\frac{2}{(s^2+s+2)}$$

Plotting constant $K$




$G(s) = K$

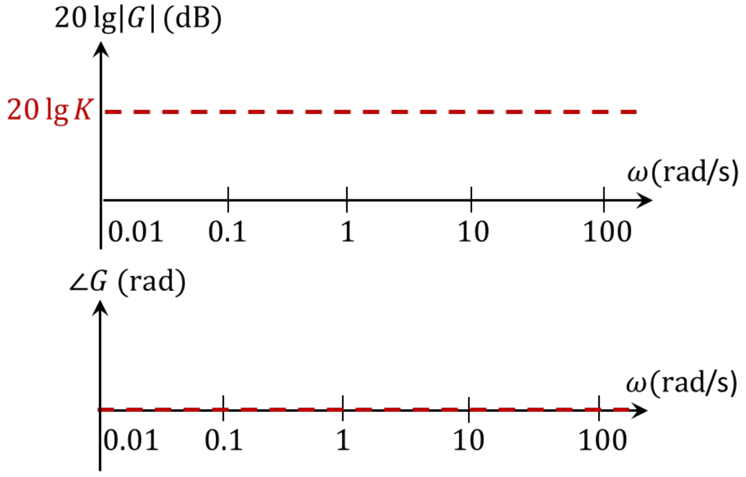
$G(s = j\omega) = Ke^{j0}$

**Magnitude**

$20\lg|G(j\omega)| = 20\lg K$

**Phase**

$\angle G(j\omega) = 0$



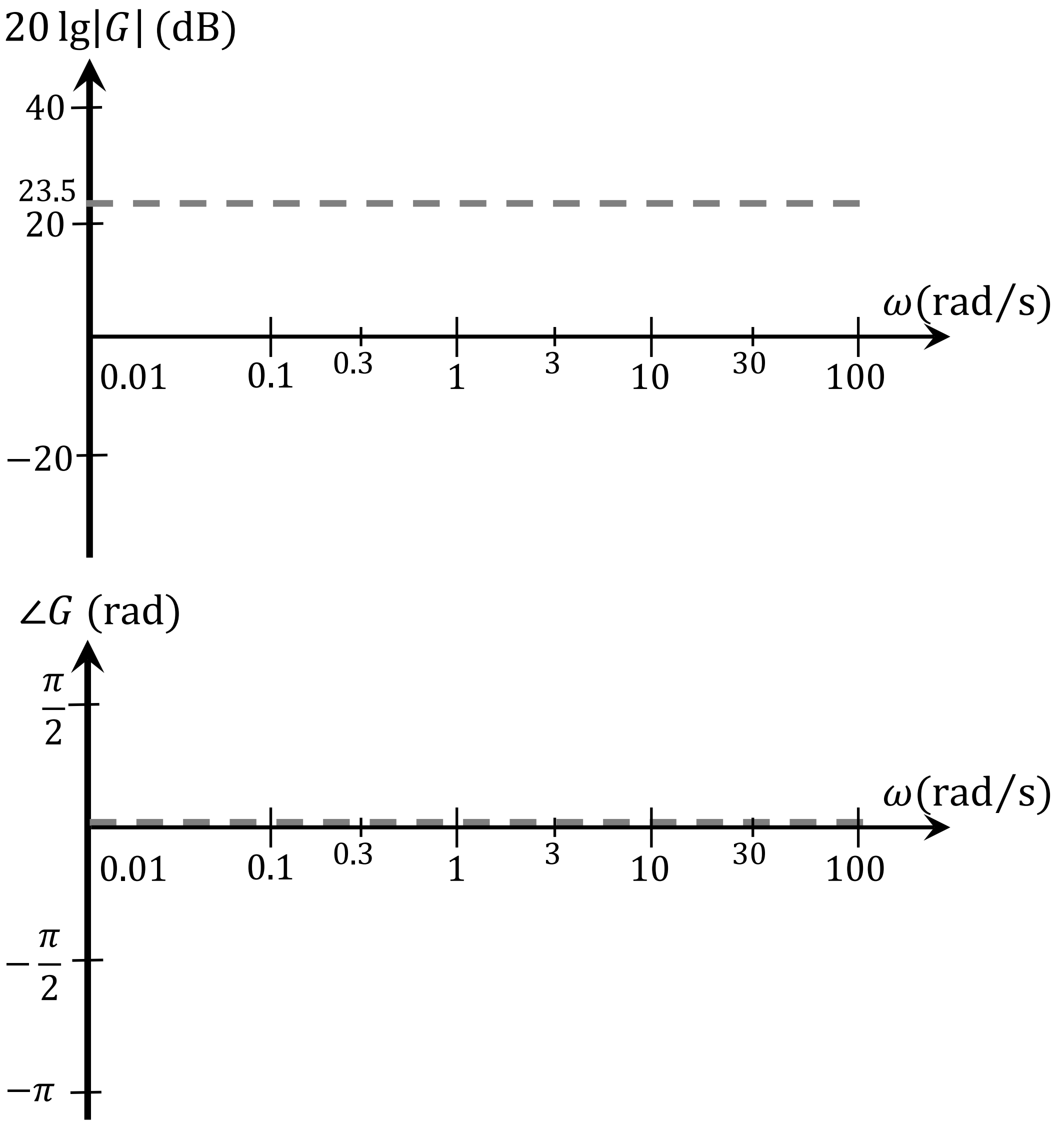

Plotting 15

<u>Magnitude plot</u>

$$20\lg G = 20\lg 15 = 23.5$$

<u>Phase plot</u>

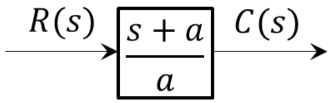
$$\angle G(j\omega) = 0$$



# Question 1

$$G(s) = 15\left(\frac{s+3}{3}\right)\left(\frac{1}{s}\right)\frac{2}{(s^2+s+2)}$$

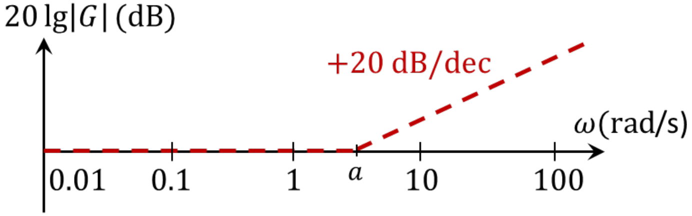
Plotting first order zero $\dfrac{s+a}{a}$




**Magnitude**

For $\omega \ll a$, $\quad 20\lg|G(j\omega)| = 0$

For $\omega \gg a$, $\quad 20\lg|G(j\omega)| = 20\lg\omega - 20\lg a$

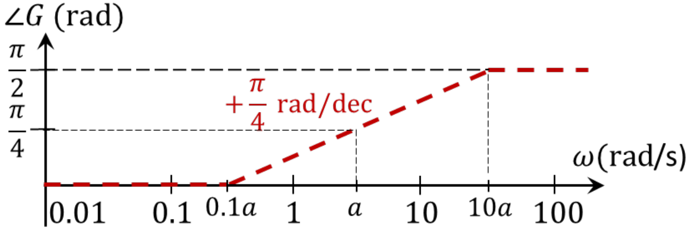



**Phase**

For $\omega < 0.1a$, $\quad \angle G(j\omega) = 0$

For $\omega = a$, $\quad \angle G(j\omega) = \pi/4$

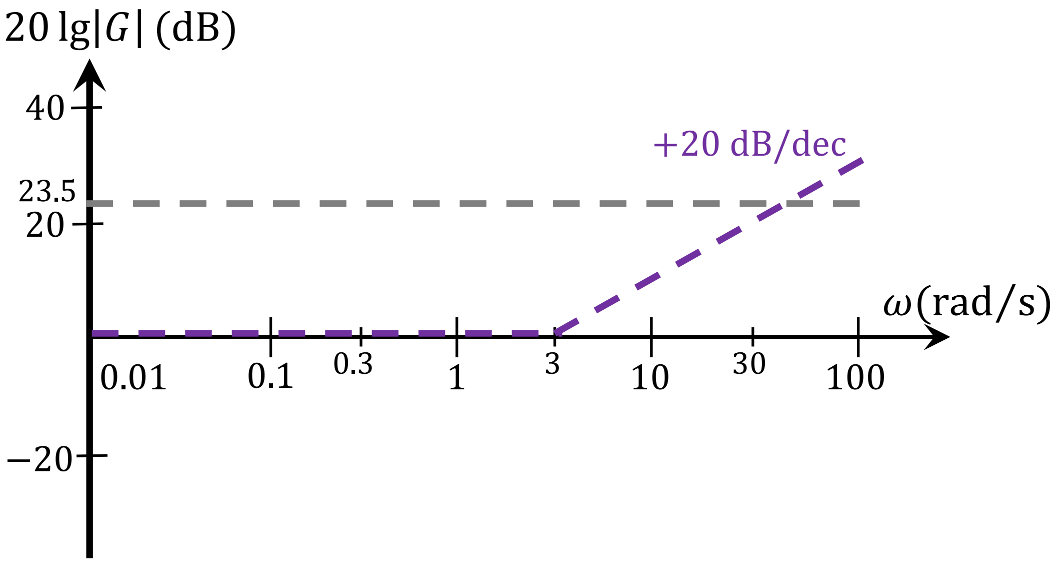
For $\omega > 10a$, $\quad \angle G(j\omega) = \pi/2$

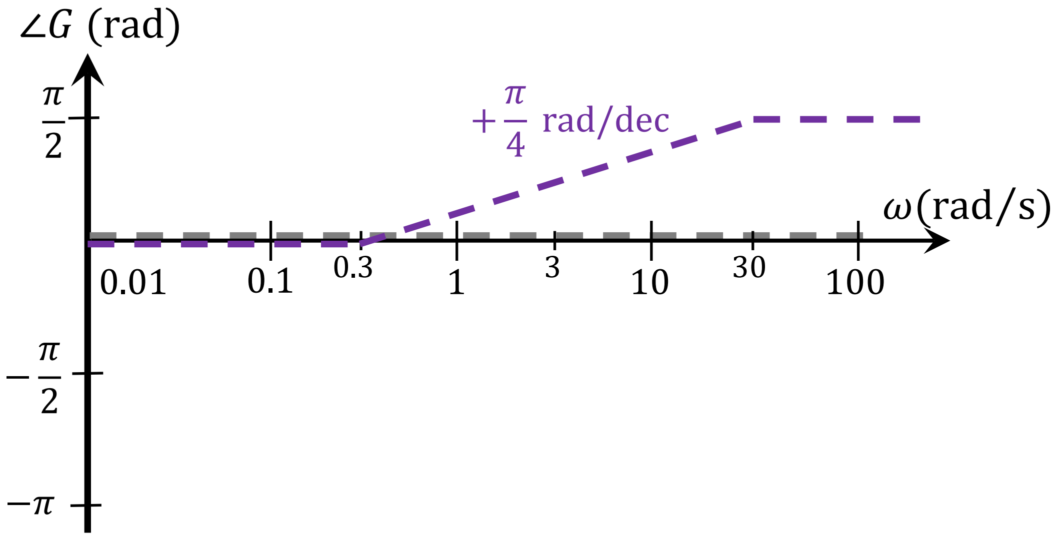



Plotting $\dfrac{s+3}{3}$

$a = 3$

Corner frequencies: $0.3, 3, 30$



# Question 1

$$G(s) = 15\left(\frac{s+3}{3}\right)\left(\frac{1}{s}\right)\frac{2}{(s^2+s+2)}$$

Plotting integrator $\frac{1}{s}$

$R(s) \rightarrow \boxed{\frac{1}{s}} \rightarrow C(s)$

$$G(s) = \frac{1}{s}$$

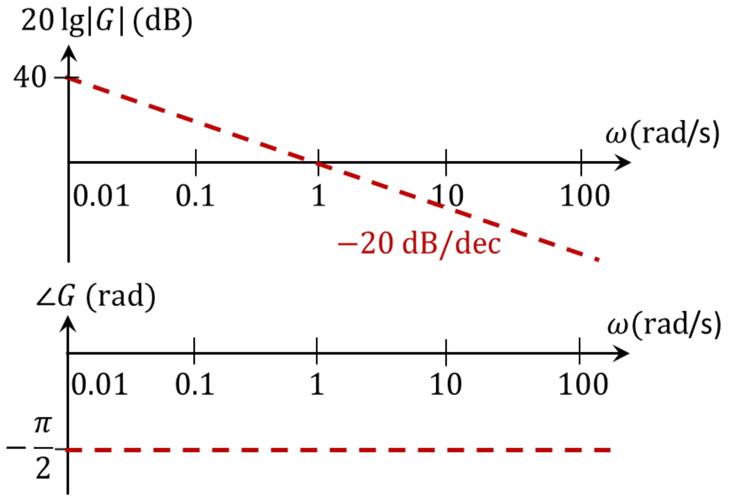
$$G(s = j\omega) = \frac{1}{j\omega}$$

**Magnitude**

$$20\lg|G(j\omega)| = -20\lg\omega$$

**Phase**

$$\angle G(j\omega) = -\frac{\pi}{2}$$



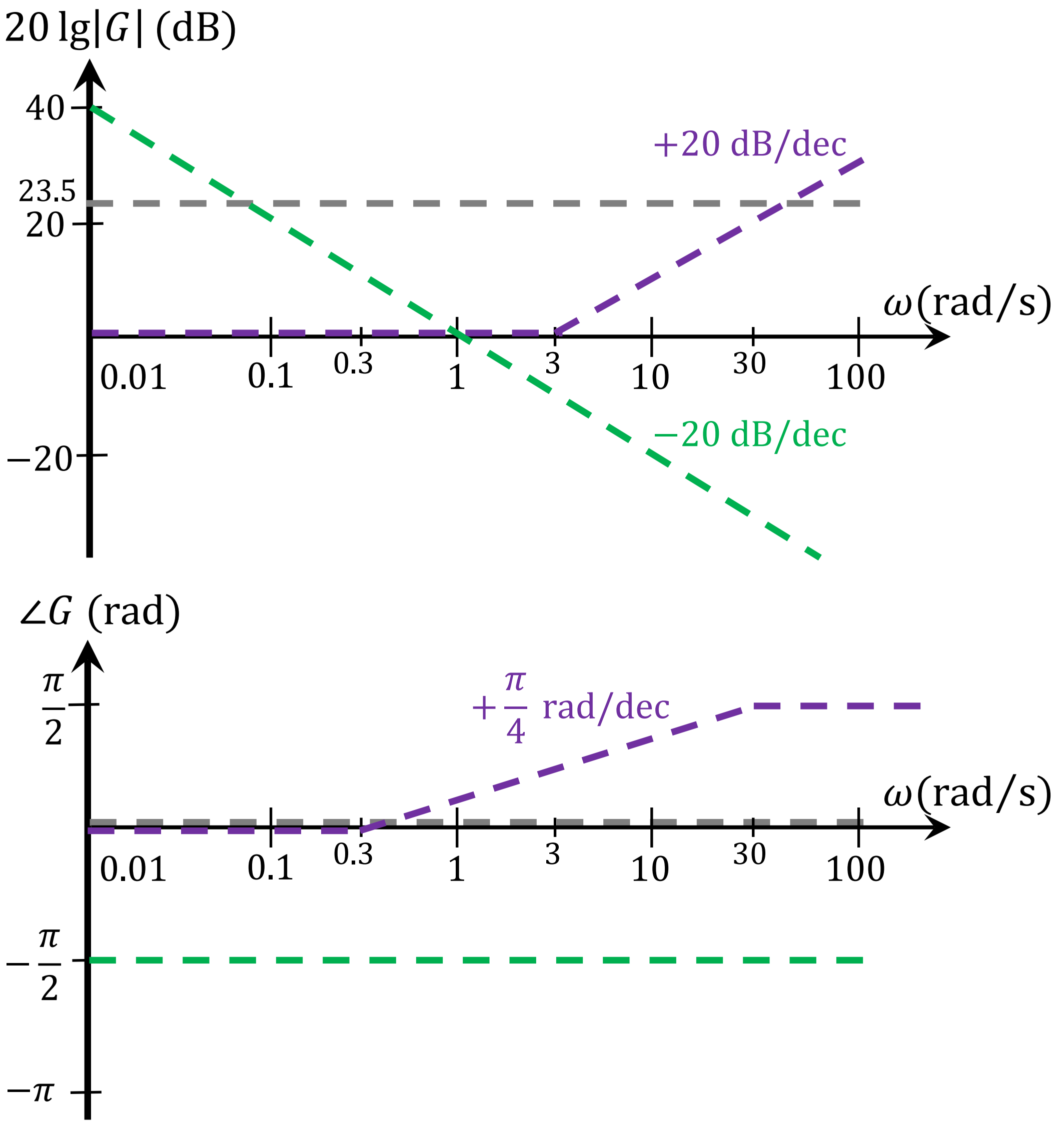

Plotting $\frac{1}{s}$

<u>Magnitude plot</u>

Line with slope $-20$ dB/dec

<u>Phase plot</u>

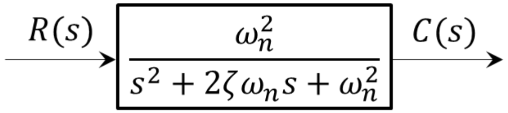
$$\angle G(j\omega) = -\frac{\pi}{2}$$



# Question 1

$$G(s) = 15\left(\frac{s+3}{3}\right)\left(\frac{1}{s}\right)\frac{2}{(s^2+s+2)}$$

Plotting second order pole $\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$




**Magnitude**

For $\omega \ll \omega_n$, $20\lg|G(j\omega)| = 0$

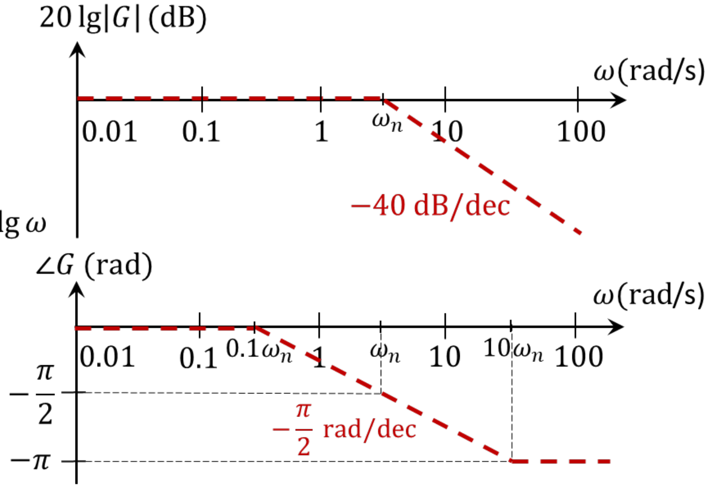
For $\omega \gg \omega_n$, $20\lg|G(j\omega)| = 40\lg\omega_n - 40\lg\omega$

**Phase**

For $\omega < 0.1\omega_n$, $\angle G(j\omega) = 0$

For $\omega = \omega_n$, $\angle G(j\omega) = -\pi/2$

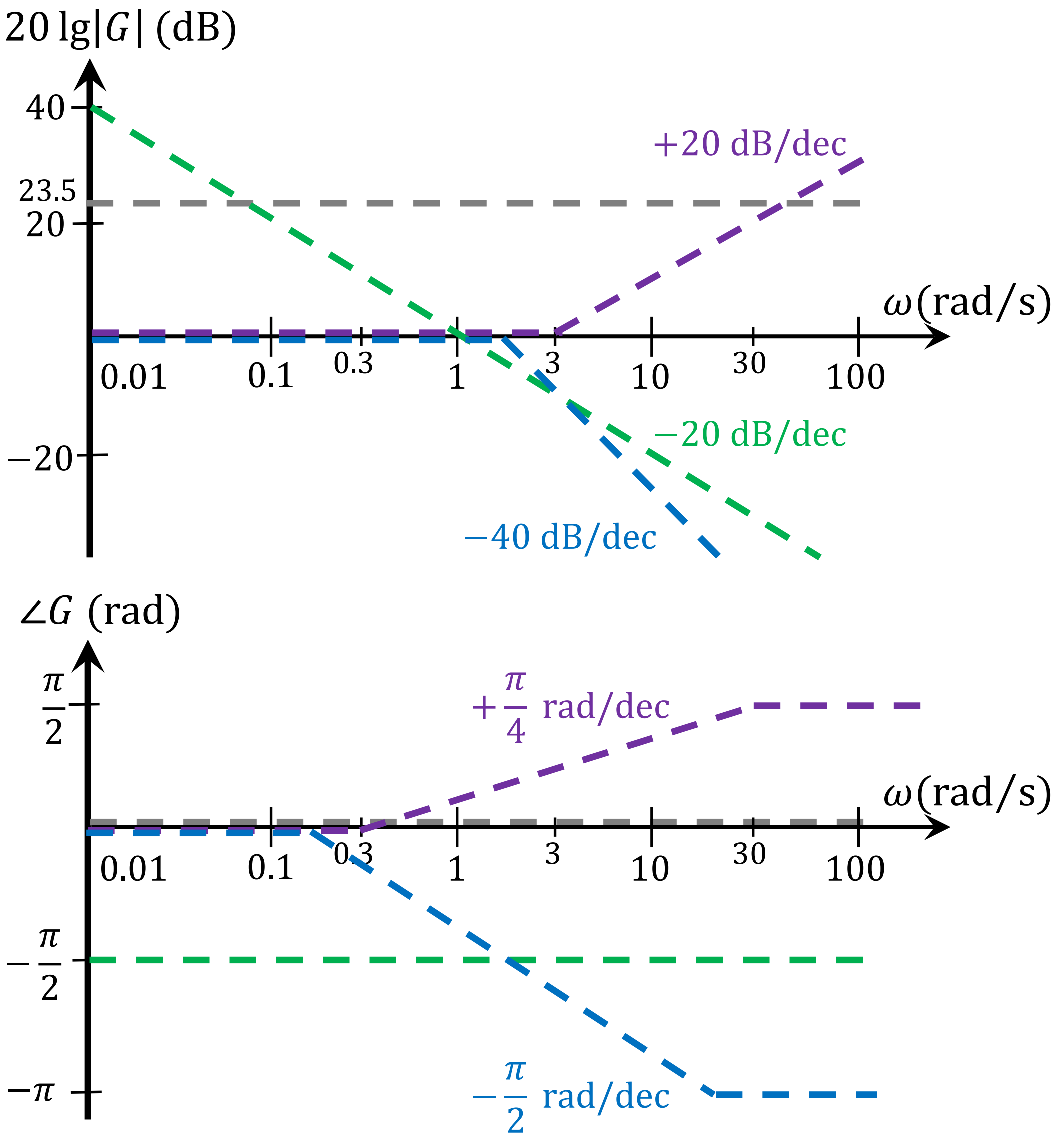
For $\omega > 10\omega_n$, $\angle G(j\omega) = -\pi$




Plotting $\dfrac{2}{(s^2+s+2)}$

$\omega_n^2 = 2$

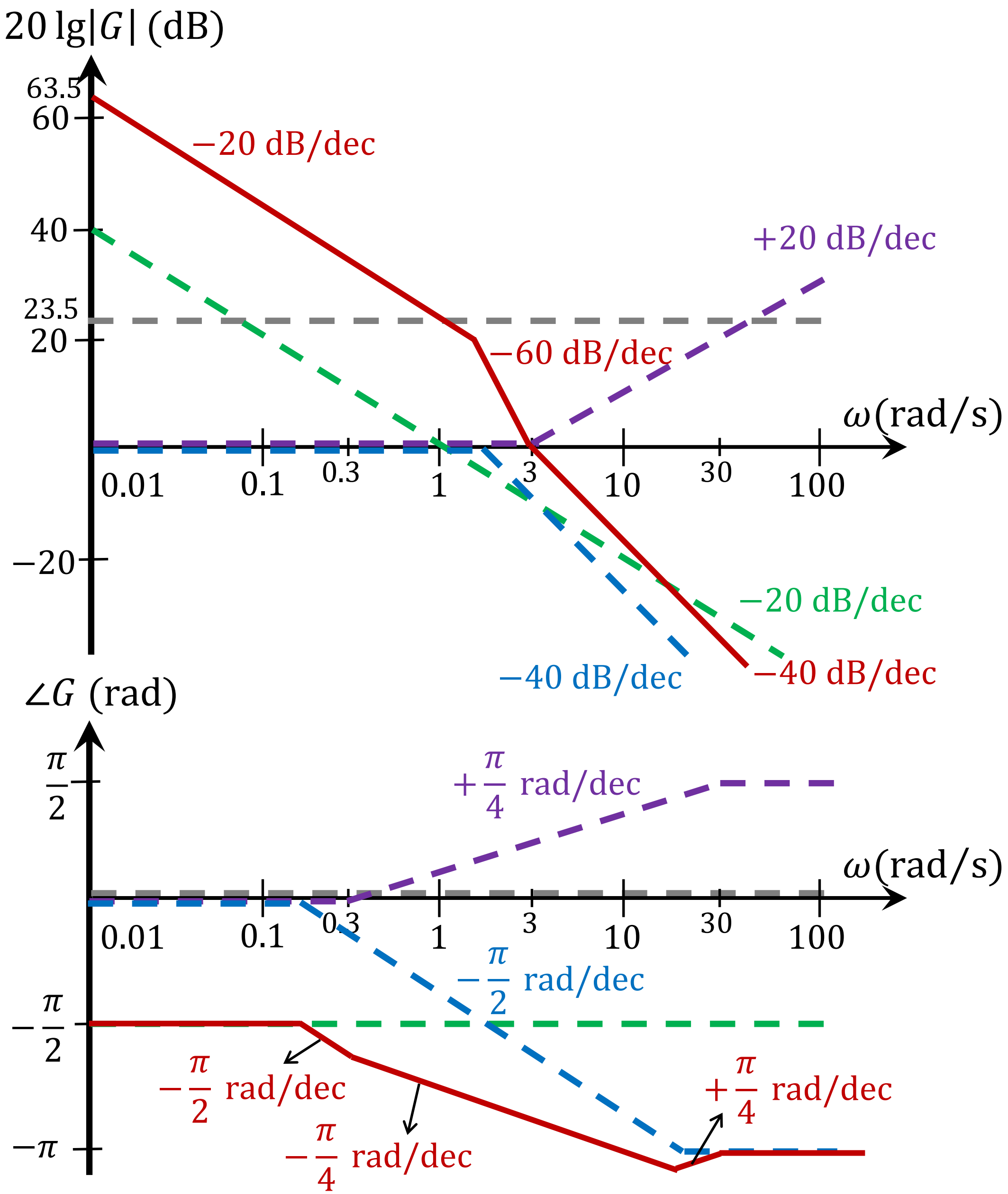
Corner frequencies: $\dfrac{\sqrt{2}}{10}, \sqrt{2}, 10\sqrt{2}$



# Question 1

$$G(s) = 15\left(\frac{s+3}{3}\right)\left(\frac{1}{s}\right)\frac{2}{(s^2+s+2)}$$

- Superposition of the plots



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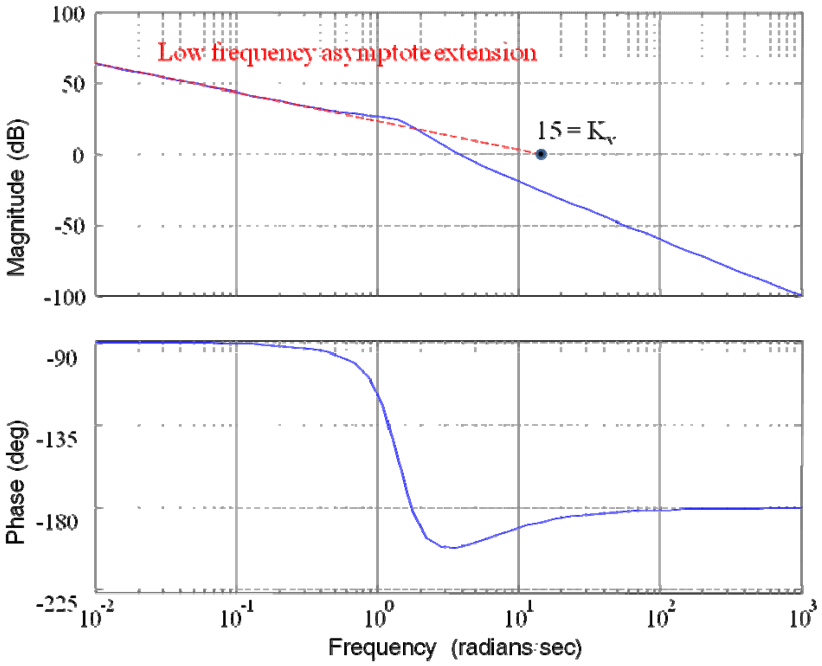


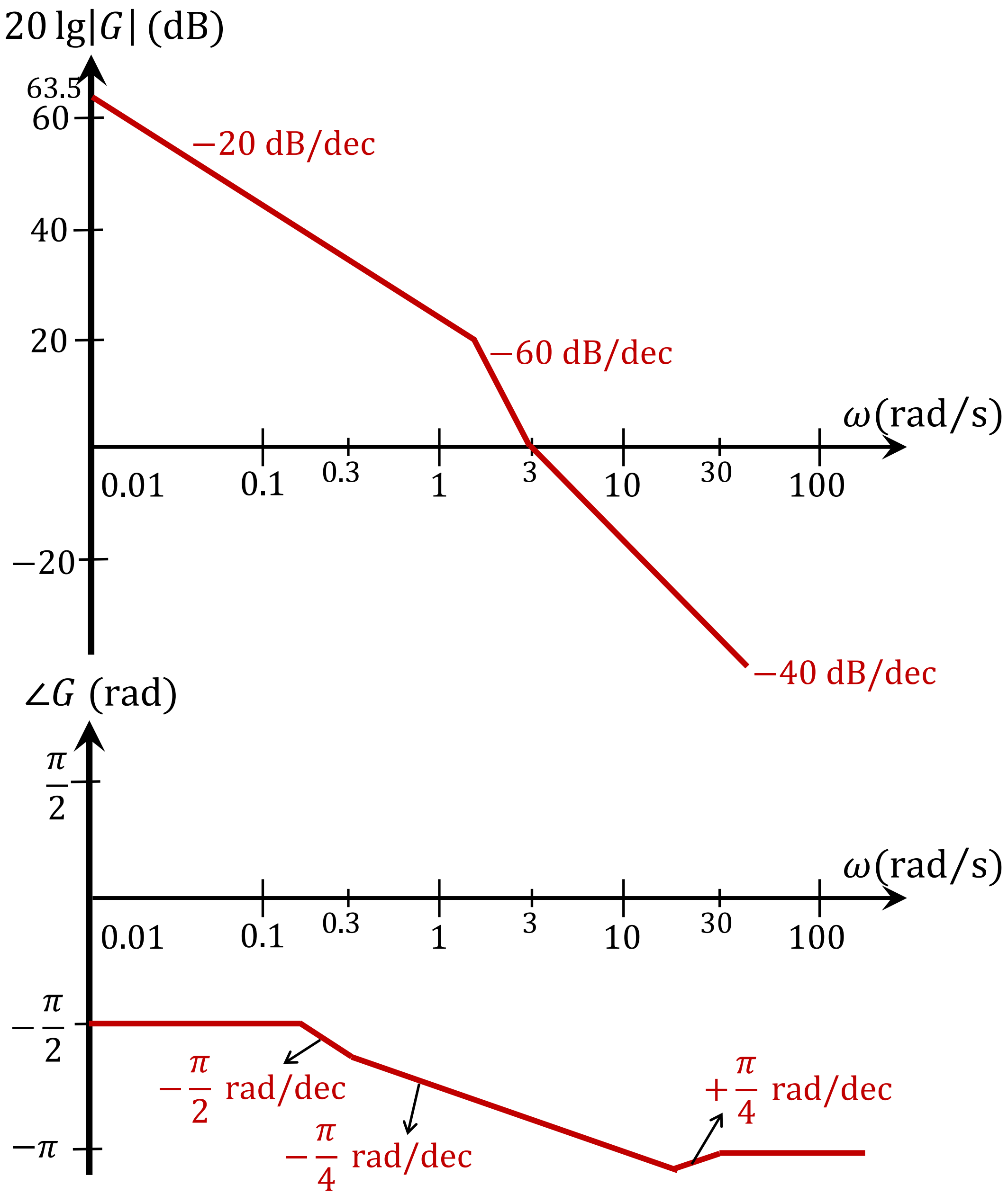

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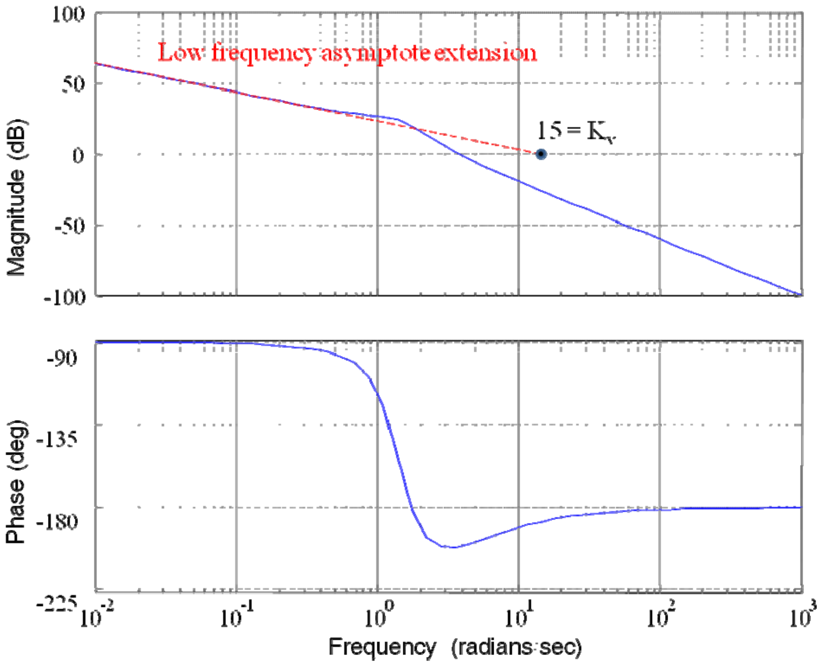
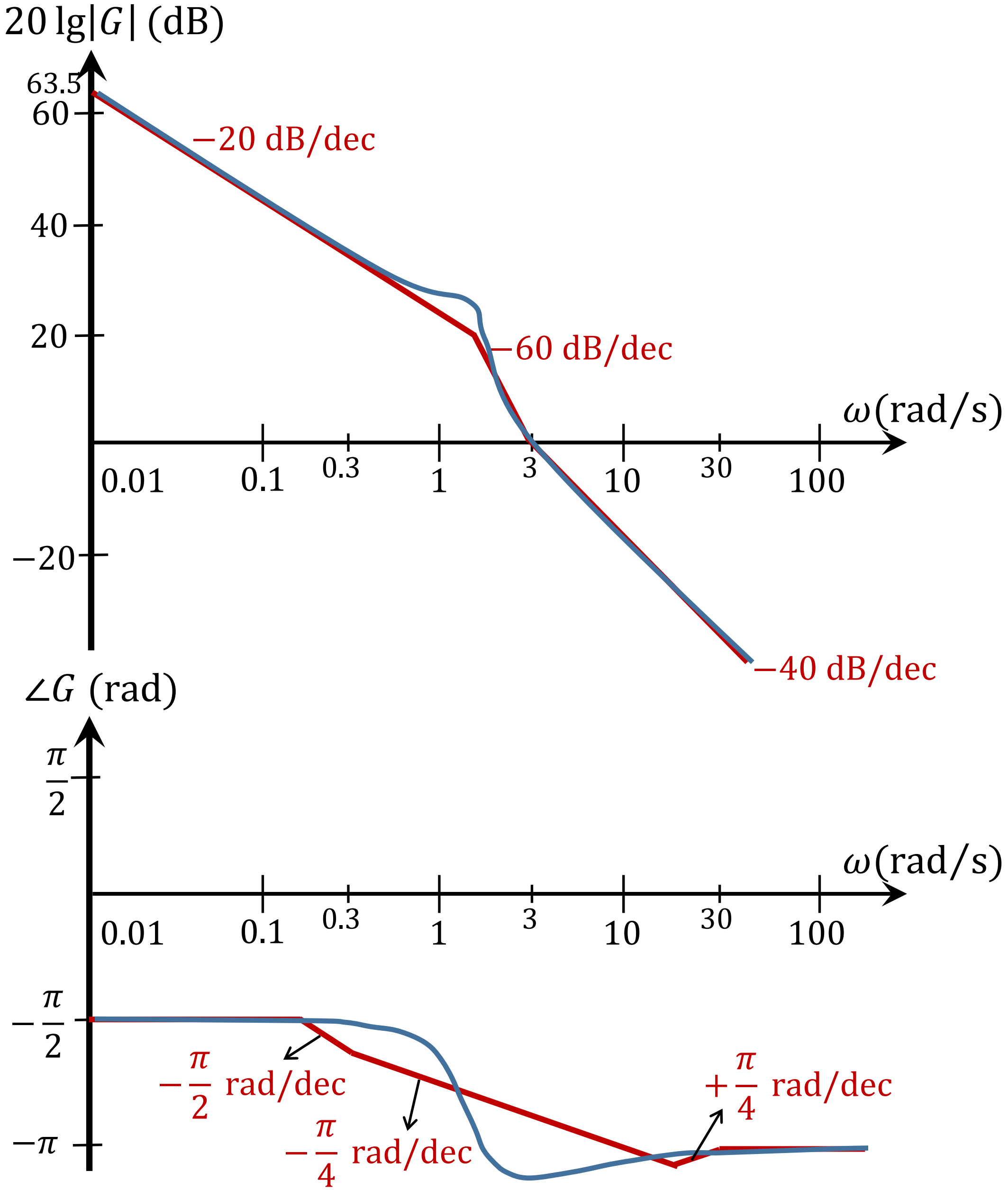
- Superposition of the plots



# Additional Notes

$$G(s) = \frac{10(s+3)}{s(s^2+s+2)}$$

**Finding the magnitude and phase equations**

$$G(j\omega) = \frac{10(j\omega+3)}{j\omega((j\omega)^2+j\omega+2)} = \frac{10(j\omega+3)}{j\omega(-\omega^2+j\omega+2)} = \frac{10\left(\sqrt{\omega^2+3^2}\, e^{j\tan^{-1}(\omega/3)}\right)}{\omega e^{j\left(\frac{\pi}{2}\right)}\left(\sqrt{(2-\omega^2)^2+\omega^2}\, e^{j\tan^{-1}\left(\frac{\omega}{2-\omega^2}\right)}\right)}$$

$$= \underbrace{\frac{10\left(\sqrt{\omega^2+3^2}\right)}{\omega\left(\sqrt{(2-\omega^2)^2+\omega^2}\right)}}_{|G(s=j\omega)|} e^{j\overbrace{\left[\tan^{-1}(\omega/3)-\frac{\pi}{2}-\tan^{-1}\left(\frac{\omega}{2-\omega^2}\right)\right]}}$$

$$\angle G(s=j\omega) = \tan^{-1}(\omega/3)-\frac{\pi}{2}-\tan^{-1}\left(\frac{\omega}{2-\omega^2}\right)$$

In dB:

$$20\lg|G(s=j\omega)| = 20\lg\left(\frac{10\left(\sqrt{\omega^2+3^2}\right)}{\omega\left(\sqrt{(2-\omega^2)^2+\omega^2}\right)}\right) = 20\lg(10) + 20\lg\left(\sqrt{\omega^2+3^2}\right) - 20\lg(\omega) - 20\lg\left(\left(\sqrt{(2-\omega^2)^2+\omega^2}\right)\right)$$

# Additional Notes

Drawing the Bode plots (gain and phase shift) of $G(s) = \dfrac{10(s+3)}{s(s^2+s+2)}$ on the 5-cycle semi-log paper.

[Here, students are required only to calculate the magnitude (dB) and phase (deg) and then, plot them on the 5-cycle semi-log paper]

a) For frequency response method, substitute $s = j\omega$ into $G(s)$: $G(j\omega) = \dfrac{10(j\omega+3)}{j\omega((j\omega)^2+j\omega+2)} = \dfrac{10(j\omega+3)}{j\omega(j\omega+(2-\omega^2))}$

b) Write down the magnitude and phase equations

The magnitude of $G$ is given by $$|G| = \frac{10\sqrt{\omega^2+3^2}}{(\omega)\left(\sqrt{(2-\omega^2)^2+\omega^2}\right)}$$

In dB: $$20\lg|G| = 20 + 20\lg\left(\sqrt{\omega^2+3^2}\right) - 20\lg(\omega) - 20\lg\left(\sqrt{(2-\omega^2)^2+\omega^2}\right)$$
$$= 20 + 20\lg\left(\sqrt{\omega^2+3^2}\right) - 20\lg(\omega) - 20\lg\left(\sqrt{\omega^4-3\omega^2+4}\right)$$

The phase of $G$ is given by $$\angle G = \tan^{-1}\left(\frac{\omega}{3}\right) - 90° - \tan^{-1}\left(\frac{\omega}{2-\omega^2}\right) \qquad \text{for } \omega < \sqrt{2}$$
$$\angle G = \tan^{-1}\left(\frac{\omega}{3}\right) - 90° - \left(180° - \tan^{-1}\left(\frac{\omega}{\omega^2-2}\right)\right) \qquad \text{for } \omega > \sqrt{2}$$

c) Identify the finite corner frequencies and arranged in ascending order: $\omega = \sqrt{2}$ and $\omega = 3$

# Additional Notes

Drawing the Bode plots (gain and phase shift) of $G(s) = \dfrac{10(s+3)}{s(s^2+s+2)}$ on the 5-cycle semi-log paper.

d) Choose the lower frequency limit for the plotting range as 2 decades below $\sqrt{2}$ i.e. $\omega_{lo} = 10^{-2}$. Based on a 5-cycle paper, the higher frequency limit $\omega_{hi}$ is $10^3 \Rightarrow \omega \in [0.01, 1000]$ rad/s.

e)
$$|G|_{\text{dB}} = 20 + 20\lg\left(\sqrt{\omega^2+3^2}\right) - 20\lg(\omega) - 20\lg\left(\sqrt{\omega^4-3\omega^2+4}\right)$$

$$\angle G = \tan^{-1}\left(\frac{\omega}{3}\right) - 90° - \tan^{-1}\left(\frac{\omega}{2-\omega^2}\right) \quad \text{for } \omega < \sqrt{2}$$

$$\angle G = \tan^{-1}\left(\frac{\omega}{3}\right) - 90° - \left(180° - \tan^{-1}\left(\frac{\omega}{\omega^2-2}\right)\right) \quad \text{for } \omega > \sqrt{2}$$

Compute $|G|_{\text{dB}}$ and $\angle G°$ for $\omega$

- For $|G|_{\text{dB}}$: more points around the corner frequencies
- For $\angle G°$: more points around the 0.1 times and 10 times the respective corner frequencies
- Sketch the Bode plots on a 5 cycle semi-log paper.

Computed values of $G(s)$ around the corner or break frequencies

| $\omega$ rad/s | $\lvert G\rvert_{\text{dB}}$ | $\angle G°$ |
|---|---|---|
| **0.01** | **64.0** | **-90** |
| **0.14** | 40.7 | -91 |
| **0.3** | 34.3 | -93 |
| 0.6 | 29.3 | -98 |
| 1.0 | 27.0 | -116 |
| **1.4** | 24.5 | -153 |
| 2.0 | 16.1 | -191 |
| **3.0** | 5.4 | -210.8 |
| 6.0 | -9.8 | -196.5 |
| 10.0 | -19.5 | -190 |
| **14.0** | -25.6 | -188 |
| 20.0 | -31.9 | -185.6 |
| **30.0** | -39.0 | -183.8 |
| **1000** | **-100** | **-180** |

Critical plotting points (in bold)

# Additional Notes

Drawing the Bode plots (gain and phase shift) of $G(s) = \dfrac{10(s+3)}{s(s^2+s+2)}$ on the 5-cycle semi-log paper.

Asymptotes

Useful for system identification and for assistance in the plotting the magnitude dB plot

- Low frequency asymptote is given by system type i.e. $G(s)$ is Type 1
- High frequency asymptote is given by system relative order i.e. [no. of poles of $G$ − no. of zeros of $G$] $= -2$.

i) Magnitude plot

- Low frequency asymptote has a gradient of $(1)(-20) = -20$ dB/dec (system type 1 due to $1/s$) between $|Gj(\omega{=}0.01)| = 64$ dB and the first corner frequency at $\omega = \sqrt{2} = 1.4$
- High frequency asymptote has a gradient of $(-2)(-20) = -40$ dB/dec between $|Gj(\omega{=}1000)| = 100$ dB and the last corner frequency $\omega = 3$

ii) Phase plot

- Low frequency asymptote has gradient $= 0$ at $\angle G(\omega{=}0.01) = -90°$ to 0.1 of the first corner frequency i.e. $\omega = 0.14$
- High frequency asymptote has gradient $= 0$ at $\angle G(\omega{=}1000) = -2 \times 90° = -180°$ to 10 times the last corner frequency i.e. $\omega = 30$.

Bode plots