NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

MA3005 Control Theory

Tutorial 10

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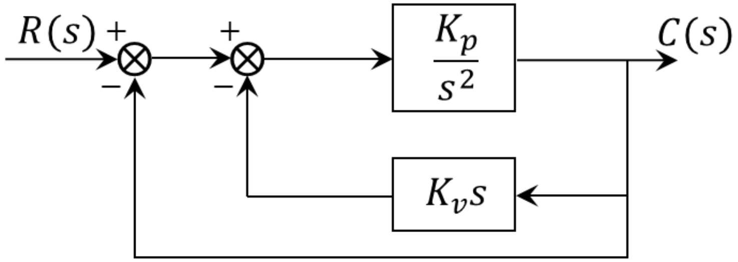
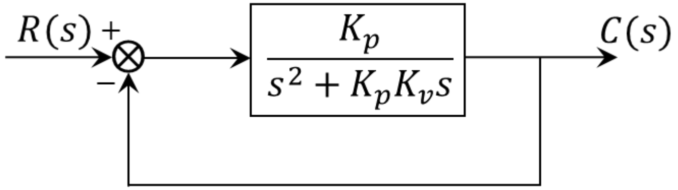
Question 1

Consider the following model of a spacecraft (same as in Tutorial 9) with a Proportional-Velocity (PV) controller

1. Find the gains K_p and K_v that yield a pair of dominant closed-loop poles at $s = -1 \pm j$.
2. Compare the step responses of the PV controller and of the PD controller designed in Tutorial 9.



$$TF = \frac{\frac{K_p}{s^2}}{1 + \frac{K_p}{s^2} K_v s} = \frac{\frac{K_p}{s^2}}{\frac{s^2 + K_p K_v s}{s^2}} = \frac{K_p}{s^2 + K_p K_v s}$$



1\.

CLTF: $$\frac{C(s)}{R(s)} = \frac{\frac{K_p}{s^2 + K_p K_v s}}{1 + \frac{K_p}{s^2 + K_p K_v s}} = \frac{K_p}{s^2 + K_p K_v s + K_p}$$

CE: $$s^2 + K_p K_v s + K_p = 0$$

Subst $s = -1 + j$: $$(-1 + j)^2 + K_p K_v(-1 + j) + K_p = 0$$

$$-2j - K_p K_v + K_p K_v j + K_p = 0$$

$$-K_p K_v + K_p + (K_p K_v - 2)j = 0$$

$$\Rightarrow -K_p K_v + K_p = 0 \qquad \Rightarrow K_p K_v - 2 = 0$$

$$K_p = 2 \qquad K_p K_v = 2$$

$$K_v = 1$$

$$\frac{C(s)}{R(s)} = \frac{2}{s^2 + 2s + 2}$$

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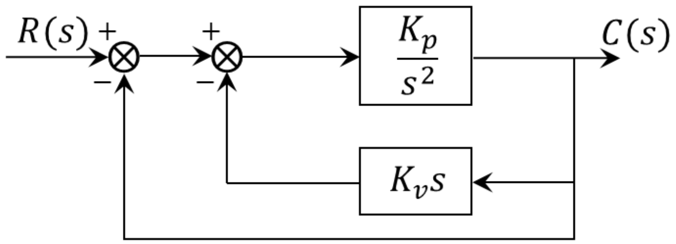
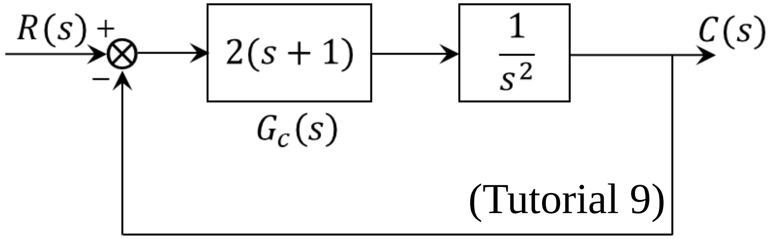


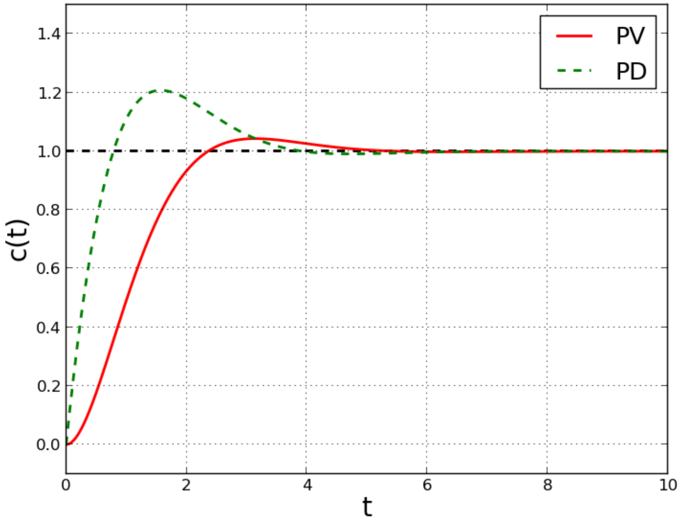
2. With PV Controller, $K_v s = s$: $\frac{C(s)}{R(s)} = \frac{2}{s^2 + 2s + 2}$

With PD Controller, $G_c(s) = K(s + z_c) = 2(s + 1)$:

$$\frac{C(s)}{R(s)} = \frac{2(s+1)}{s^2 + 2s + 2}$$



There's an "s" term in the numerator ⇒ Not a "pure" 2nd order system



With PV Controller

Rise Time: 1.52
Settling Time: 4.22
Overshoot: 4.32
Peak: 1.04
Peak Time: 3.13

With PD Controller

Rise Time: 0.60
Settling Time: 3.46
Overshoot: 20.79
Peak: 1.21
Peak Time: 1.57

⇒ Different transient response with similar settling time
⇒ PD may increase overshoot

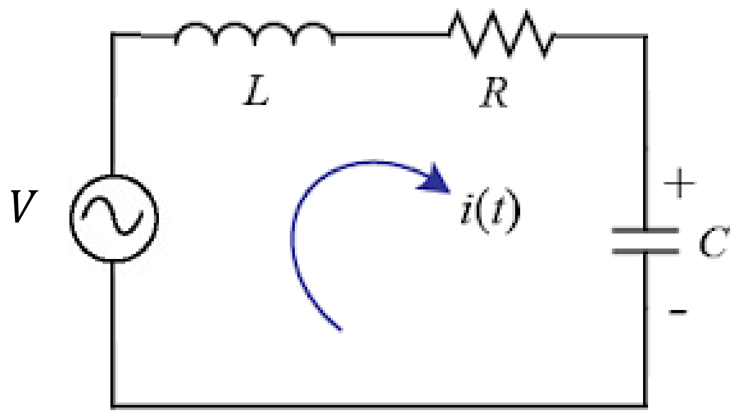
MATLAB script:

```
PV1=2;PV2=[1 2 2];
PD1=[2 2];PD2=[1 2 2];
PV_sys=tf(PV1,PV2);
PD_sys=tf(PD1,PD2);

time=0:0.1:10;
PV=step(PV_sys,time);
PD=step(PD_sys,time);
PV_info = stepinfo(PV_sys);
PD_info = stepinfo(PD_sys);

figure
plot(time,PV,time,PD);
```

Background on Electrical and Electronic Systems



$$V_L(t) + V_R(t) + V_C(t) = V(t)$$

$$L\frac{di(t)}{dt} + R\, i(t) + \frac{1}{C}\int i\, dt = V(t)$$

For resistor:

$$V_R(t) = R\, i(t)$$
$$V_R(s) = R\, I(s)$$
$$V_R(j\omega) = R\, I(j\omega)$$

For inductor:

$$V_L(t) = L\frac{di(t)}{dt}$$
$$V_L(s) = Ls\, I(s)$$
$$V_L(j\omega) = \boxed{j\omega L} I(j\omega)$$

"resistance" of inductor

For capacitor:

$$V_C(t) = \frac{1}{C}\int i\, dt$$
$$\frac{dV_C(t)}{dt} = \frac{1}{C} i(t)$$
$$V_C(s) = \frac{1}{Cs} I(s)$$
$$V_C(j\omega) = \boxed{\frac{1}{j\omega C}} I(j\omega)$$

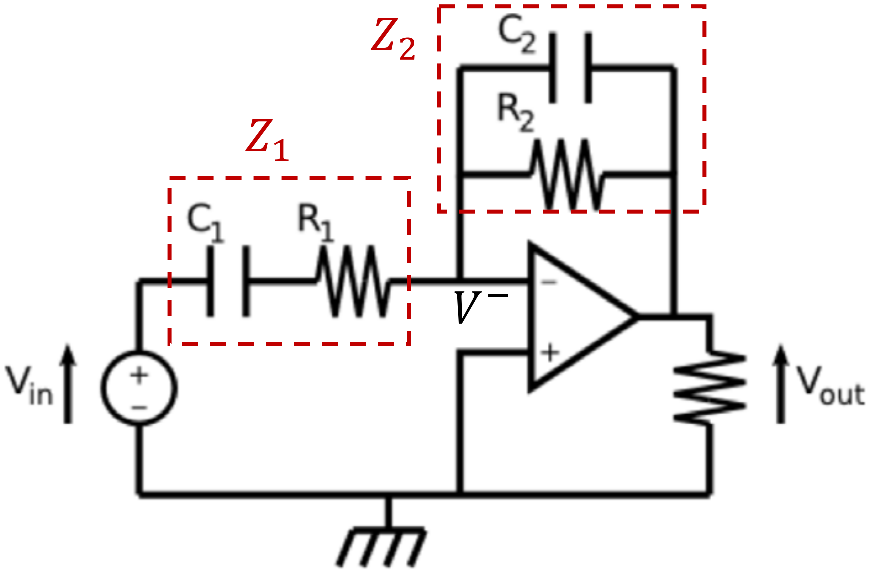
"resistance" of capacitor

Impedance is the frequency domain ratio of the voltage to the current

$$\boxed{\begin{aligned} Z_R(j\omega) &= R \\ Z_L(j\omega) &= j\omega L \\ Z_C(j\omega) &= \frac{1}{j\omega C} \end{aligned}}$$

Question 2

Consider the following circuit:



1. Show that the circuit has the following transfer function

$$\frac{V_{out}}{V_{in}} = -\frac{R_2}{R_1}\frac{j\frac{\omega}{\omega_1}}{(1+j\frac{\omega}{\omega_1})(1+j\frac{\omega}{\omega_2})} \quad \text{where } \omega_1 = \frac{1}{R_1C_1} \text{ and } \omega_2 = \frac{1}{R_2C_2}$$

$$\boxed{\begin{aligned} Z_R(j\omega) &= R \\ Z_L(j\omega) &= j\omega L \\ Z_C(j\omega) &= \frac{1}{j\omega C} \end{aligned}}$$

$$\begin{aligned} Z_1 &= R_1 + \frac{1}{j\omega C_1} \\ &= R_1 + \frac{R_1}{j\omega R_1 C_1} \\ &= \frac{R_1\, j\omega R_1 C_1}{j\omega R_1 C_1} + \frac{R_1}{j\omega R_1 C_1} \\ &= \frac{R_1}{j\omega R_1 C_1}(j\omega R_1 C_1 + 1) \\ &= \frac{R_1(1+j\frac{\omega}{\omega_1})}{j\frac{\omega}{\omega_1}} \end{aligned}$$

$$Z_2 = R_2 \,//\, \frac{1}{j\omega C_2}$$

$$\frac{1}{Z_2} = \frac{1}{R_2} + \frac{j\omega C_2}{1}$$

$$\Rightarrow Z_2 = \frac{R_2}{1+j\omega C_2 R_2} = \frac{R_2}{1+j\frac{\omega}{\omega_2}}$$

Note: In ideal op amp, no current flows into the input terminals

$$i^- = 0 \quad \rightarrow \frac{V_{in} - V^-}{Z_1} = \frac{V^- - V_{out}}{Z_2}$$

$$V^- = 0 \quad \rightarrow \frac{V_{in}}{Z_1} = \frac{-V_{out}}{Z_2}$$

$$\Rightarrow \frac{V_{out}}{V_{in}} = \frac{-Z_2}{Z_1} = \frac{-R_2}{R_1}\frac{(j\frac{\omega}{\omega_1})}{(1+j\frac{\omega}{\omega_1})(1+j\frac{\omega}{\omega_2})}$$

Question 2

2. From now on, assume the following numerical values: $R_1 = 1\Omega, R_2 = 10\Omega, C_1 = 1mF, C_2 = 1\mu F$. Determine the amplitude and the phase of the transfer function in the following regimes
   - $\omega \ll \omega_1$
   - $\omega_1 \ll \omega \ll \omega_2$
   - $\omega_2 \ll \omega$

$$\boxed{\text{Let } G(s=j\omega) = \frac{V_{out}}{V_{in}} = \frac{-R_2}{R_1}\frac{j\frac{\omega}{\omega_1}}{(1+j\frac{\omega}{\omega_1})(1+j\frac{\omega}{\omega_2})}}$$

Find:

$|G(s=j\omega)|$ $\quad \angle G(s=j\omega)$

- $\omega \ll \omega_1$:

$$G(s=j\omega) = -\frac{R_2}{R_1}\frac{j\frac{\omega}{\omega_1}}{(1+j0)(1+j0)}$$

$$= -\frac{R_2}{R_1}\left[\frac{j\frac{\omega}{\omega_1}}{1}\right]$$

$$= \frac{R_2}{R_1}e^{-j\pi}\left[\frac{\omega}{\omega_1}e^{j\frac{\pi}{2}}\right]$$

$$= \boxed{\frac{R_2}{R_1}\frac{\omega}{\omega_1}}e^{\boxed{-\frac{\pi}{2}j}}$$

$$|G(s=j\omega)| = \frac{R_2}{R_1}\frac{\omega}{\omega_1}(\approx 0) \qquad \angle G(s=j\omega) = -{}^{\pi}/_{2}$$

- $\omega_1 \ll \omega \ll \omega_2$:

$$G(s=j\omega) = -\frac{R_2}{R_1}\frac{j\frac{\omega}{\omega_1}}{\left(j\frac{\omega}{\omega_1}\right)(1+j0)}$$

$$= -\frac{R_2}{R_1}\left[\frac{j\frac{\omega}{\omega_1}}{j\frac{\omega}{\omega_1}(1)}\right] = -\frac{R_2}{R_1}$$

$$= \frac{R_2}{R_1}e^{-\pi j}$$

$$|G(s=j\omega)| = \frac{R_2}{R_1}$$

$$\angle G(s=j\omega) = -\pi$$

- $\omega_2 \ll \omega$:

$$G(s=j\omega) = -\frac{R_2}{R_1}\frac{j\frac{\omega}{\omega_1}}{\left(j\frac{\omega}{\omega_1}\right)\left(j\frac{\omega}{\omega_2}\right)}$$

$$= -\frac{R_2}{R_1}\left[\frac{1}{\left(j\frac{\omega}{\omega_2}\right)}\right] = \frac{R_2}{R_1}\frac{j}{\left(\frac{\omega}{\omega_2}\right)}$$

$$= \frac{R_2}{R_1\left(\frac{\omega}{\omega_2}\right)}e^{-\frac{3\pi}{2}j}$$

$$|G(s=j\omega)| = \frac{R_2}{R_1\left(\frac{\omega}{\omega_2}\right)}(\approx 0)$$

$$\angle G(s=j\omega) = -{}^{3\pi}/_{2}$$

Question 2

$R_1 = 1\Omega \quad R_2 = 10\Omega$

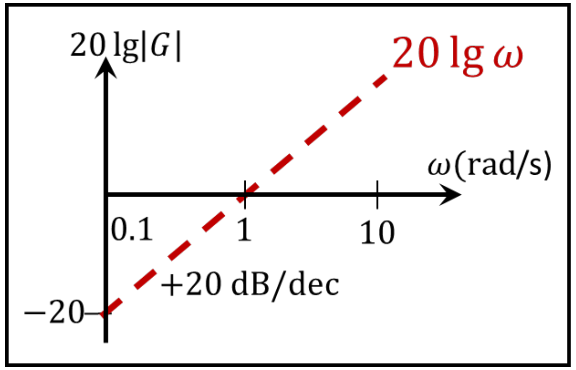
$C_1 = 1mF \quad C_2 = 1\mu F$

$$\omega_1 = \frac{1}{R_1 C_1} = 10^3$$

$$\omega_2 = \frac{1}{R_2 C_2} = 10^5$$

$$G(s = j\omega) = \frac{-R_2}{R_1} \frac{j\frac{\omega}{\omega_1}}{(1 + j\frac{\omega}{\omega_1})(1 + j\frac{\omega}{\omega_2})}$$

$$G(s = j\omega) = -10 \frac{j\frac{\omega}{10^3}}{\left(1 + j\frac{\omega}{10^3}\right)\left(1 + j\frac{\omega}{10^5}\right)}$$



- $\omega \ll 10^3$

$$|G(s = j\omega)| = \frac{R_2}{R_1}\frac{\omega}{\omega_1} = \frac{1}{10^2}\omega$$

$$20\lg|G(s = j\omega)| = -40 + 20\lg\omega$$

$$\angle G(s = j\omega) = -{}^{\pi}/_{2}$$

- $10^5 \ll \omega$

$$|G(s = j\omega)| = \frac{R_2}{R_1\left(\frac{\omega}{\omega_2}\right)} = \frac{10^6}{\omega}$$

$$20\lg|G(s = j\omega)| = 120 - 20\lg\omega$$

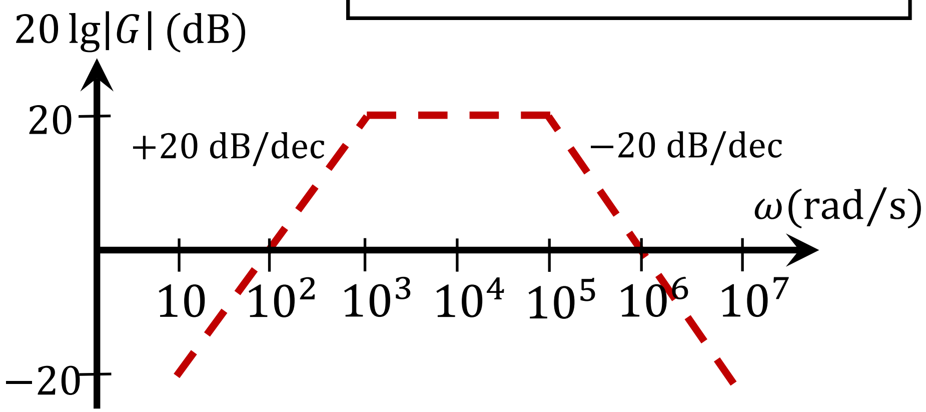
$$\angle G(s = j\omega) = -{}^{3\pi}/_{2}$$

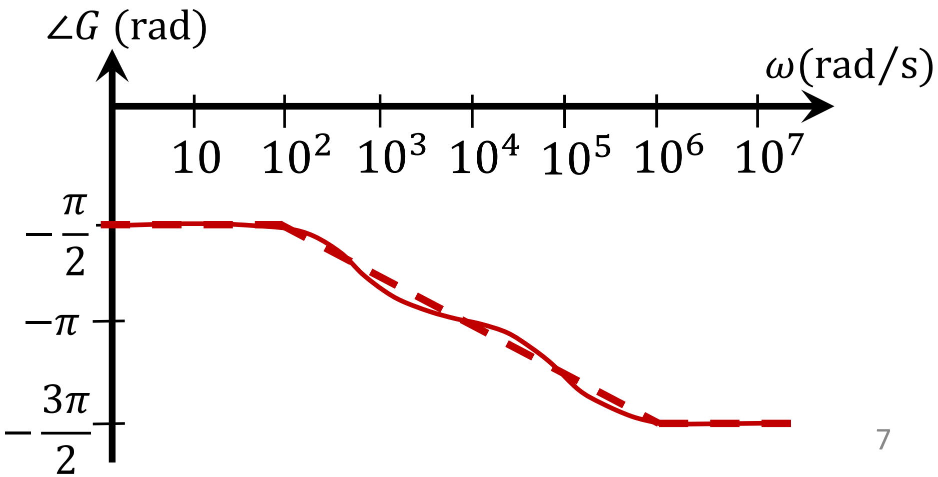
- $10^3 \ll \omega \ll 10^5$

$$|G(s = j\omega)| = \frac{R_2}{R_1}$$

$$20\lg|G(s = j\omega)| = 20\lg\frac{R_2}{R_1} = 20\lg\frac{10}{1} = 20$$

$$\angle G(s = j\omega) = -\pi$$





Question 2 EXTRA

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Drawing straight lines bode plots

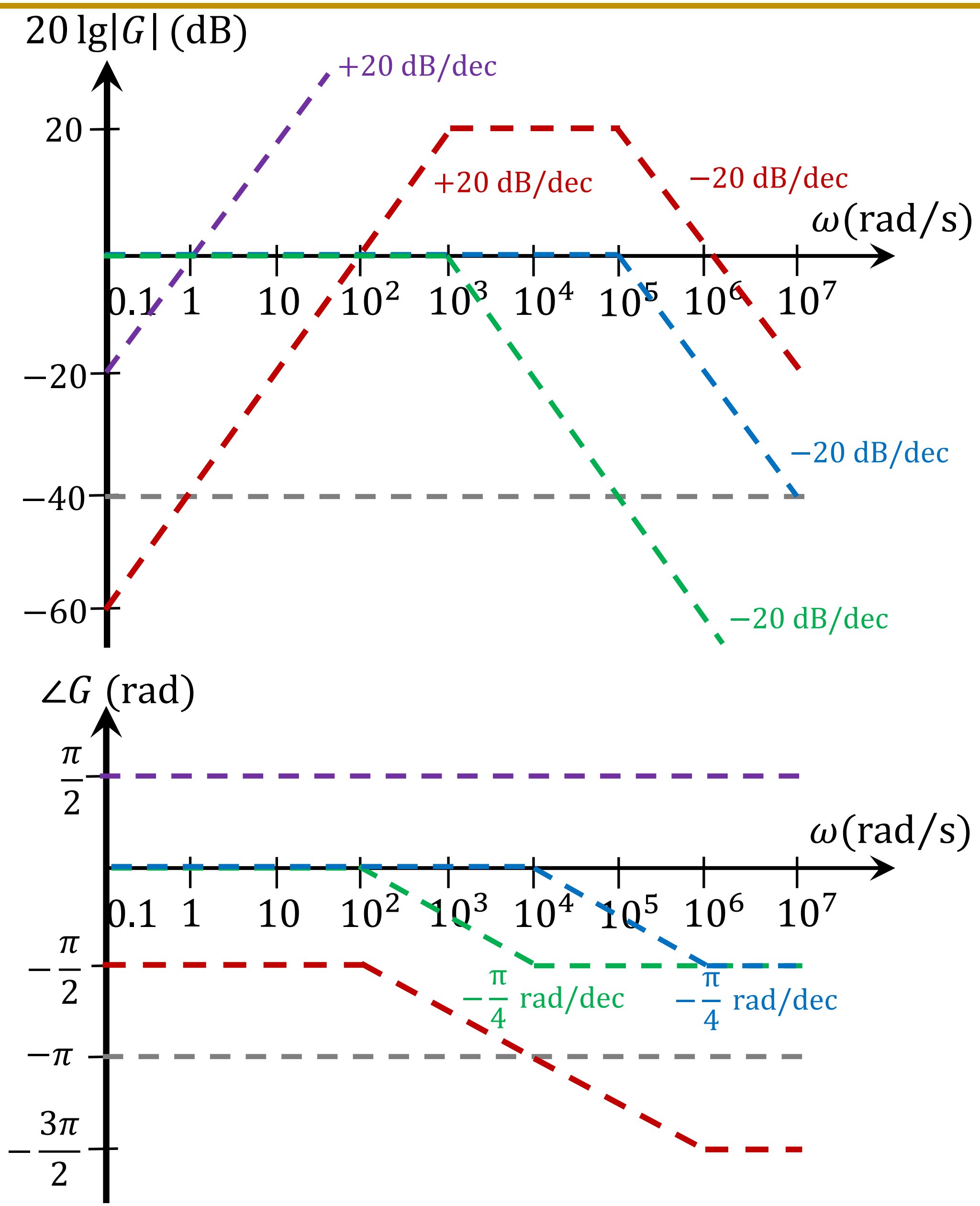
$$G(s=j\omega) = -10\frac{j\dfrac{\omega}{10^3}}{\left(1+j\dfrac{\omega}{10^3}\right)\left(1+j\dfrac{\omega}{10^5}\right)}$$

$$G(s) = -10\frac{\dfrac{s}{10^3}}{\left(1+\dfrac{s}{10^3}\right)\left(1+\dfrac{s}{10^5}\right)}$$

$$G(s) = -10\frac{\dfrac{s}{10^3}}{\left(\dfrac{s+10^3}{10^3}\right)\left(\dfrac{s+10^5}{10^5}\right)}$$

$$G(s) = -\frac{1}{10^2}(s)\left(\frac{10^3}{s+10^3}\right)\left(\frac{10^5}{s+10^5}\right)$$

Corner frequencies



Question 2

3. Determine (approximately) the responses $V_{out}(t)$ of the circuit to the following inputs
   - $V_{in}(t) = 5\,sin(10^4 t)$
   - $V_{in}(t) = 2\,cos(10t)$
   - $V_{in}(t) = 5\,sin(10^4 t) + 2\,cos(10t) - 3\,cos(10^7 t)$

Note:

$r(t) = r_0\,sin(\omega t) \qquad \frac{C}{R} = G(s)$

$c(t) = r_0|G(s = j\omega)|\,sin(\omega t + \angle G(s = j\omega))$

(steady state)

- $V_{in}(t) = 5\,sin(10^4 t)$

$\omega = 10^4 \Rightarrow \omega_1 \ll \omega \ll \omega_2$

(within passband)

$\omega_1 = 10^3$ & $\omega_2 = 10^5$

$$|G(s = j\omega)| = \frac{R_2}{R_1} = 10$$

$$\angle G(s = j\omega) = -\pi$$

can also be derived graphically using bode plot (if given)

At $\omega = 10^4$, $20\lg|G| = 20$

$\lg|G| = 1$

$\Rightarrow |G(s = 10^4 j)| = 10$

$\angle G(s = j\omega) = -\pi$

$$\Rightarrow V_{out} = V_{in_0}|G(s = j\omega)|\,sin(\omega t + \angle G(s = j\omega))$$

$$= 5(10)\,sin(10^4 t + (-\pi))$$

$$= 50\,sin(10^4 t - \pi)$$

- $V_{in}(t) = 2\,cos(10t)$

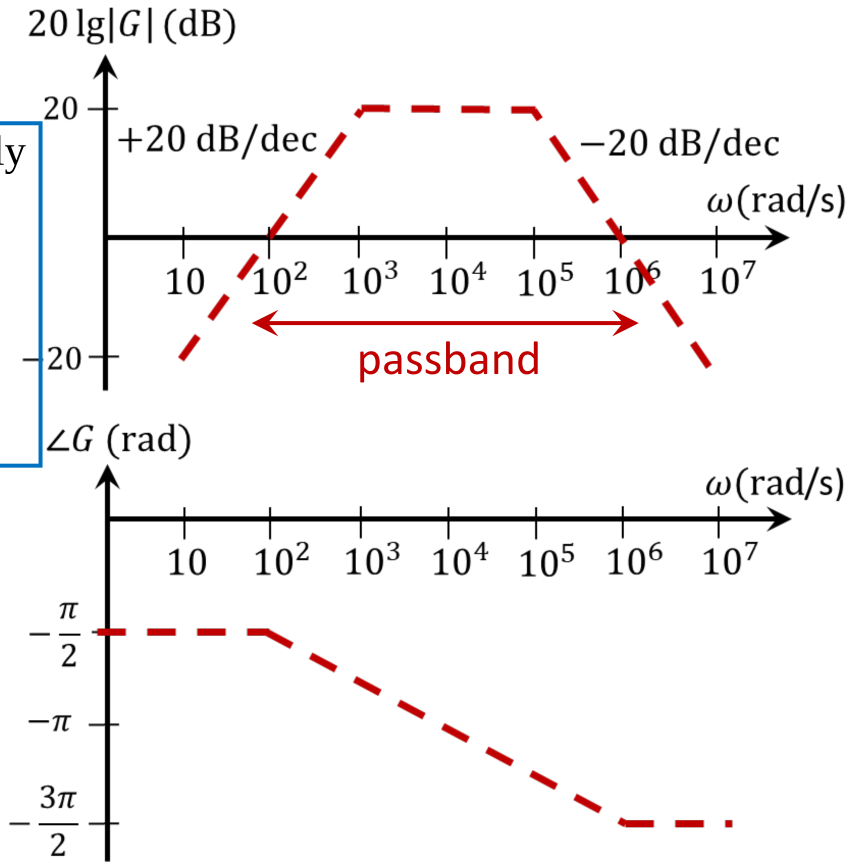
$\omega = 10 \Rightarrow \omega \ll \omega_1$

$$|G(s = j\omega)| = \frac{R_2}{R_1}\frac{\omega}{\omega_1} = 0.1$$

$$\angle G(s = j\omega) = -{}^{\pi}/_{2}$$

$$\Rightarrow V_{out} = V_{in_0}|G(s = j\omega)|\,cos(\omega t + \angle G(s = j\omega))$$

$$= 0.2\cos(10t - {}^{\pi}/_{2})$$ (attenuated)



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Note:
$r(t) = r_0\,sin(\omega t)$ $\qquad \frac{C}{R} = G(s)$
$c(t) = r_0|G(s = j\omega)|\,sin(\omega t + \angle G(s = j\omega))$
(steady state)

- $V_{in}(t) = 5\,sin(10^4 t) + 2\,cos(10t) - 3\,cos(10^7 t)$

  Only the signal at $\omega = 10^4$ will pass (the first term)

  $|G(s = j\omega)| = \frac{R_2}{R_1} = 10 \qquad \angle G(s = j\omega) = -\pi$

  $\Rightarrow V_{out} = V_{in_0}|G(s = j\omega)|\,sin(\omega t + \angle G(s = j\omega))$

  $= 50\,sin(10^4 t - \pi)$