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Notes on 2nd order systems

Consider a 2nd order system such as a linear mass-spring-damper, with mass m, linear friction coefficient b and linear stiffness constant k, for which the following dynamics (Newton's law) hold:

$$f(t) = m\ddot{x}(t) + b\dot{x}(t) + kx(t)$$

This corresponds to a 2nd order Ordinary Differential Equation (ODE)

NOTE: the **order** of an ODE is given the order of the highest time derivative. For Newton's law (F=ma) we always have the acceleration $a\equiv\ddot{x}$ which implies 2nd order dynamics.

Given a 2nd order ODE, the evolution of the system x(t) is uniquely determined if we are given 'two pieces of information', typically position x_0 and velocity $v_0 \equiv \dot{x}_0$ at a given time t_0 . This means that, at any time t, the '**state**' s(t) of a system is fully captured by a pair

$$m{x}(t) \equiv egin{bmatrix} x(t) \ v(t) \end{bmatrix}$$

where we conveniently introduce a new '**velocity**' variable $v(t) \equiv \dot{x}(t)$.

State Space representation

Noting that $\dot{v}=\ddot{x}$, Newton's law can be rewritten two equations in two variables x(t) and v(t) but involving only first derivatives (instead of the second derivatives of the original formulation):

$$egin{cases} \dot{x} &= v \ f &= m\dot{v} + bv + kx \end{cases}$$

rearranging by keeping derivatives on the left hand side

$$egin{cases} \dot{x} &= v \ \dot{v} &= rac{1}{m}(-kx-bv+f) \end{cases}$$

Or, equivalently, in matrix format

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$$rac{d}{dt}egin{bmatrix} x \ v \end{bmatrix} = \underbrace{egin{bmatrix} 0 & 1 \ -rac{k}{m} & -rac{b}{m} \end{bmatrix}}_A egin{bmatrix} x \ v \end{bmatrix} + \underbrace{egin{bmatrix} 0 \ rac{1}{m} \end{bmatrix}}_B f$$

i.e.

$$oxed{rac{d}{dt}oldsymbol{x} = Aoldsymbol{x} + Bf}$$

AC Analysis

In AC Analysis, we drive systems with purely sinusoidal inputs, at a generic (radian) frequency ω and we simply look for solutions which are also purely sinusoidal, at the same frequency ω .

$$\cos \omega t \Rightarrow \boxed{ exttt{system}} \Rightarrow A_\omega \cos(\omega t + \phi_\omega)$$

NOTE: the radian frequency ω , measured in [rad/sec], is related to the natural frequency f, measured in [Hz], simply by 2π factor, i.e.

$$\omega = 2\pi f$$

Generalized Sinusoids via Complex Exponentials

Computationally, it is more efficient to consider 'generalized' sinusoids via complex exponentials

$$e^{jx} := \cos x + j \sin x$$

where
$$j = \sqrt{-1}$$
.

NOTE: Given a generalized signal $\tilde{s}(t):=S_0e^{j\omega t}$, where $S_0\in\mathbb{C}$ is a complex number (or **phasor** of the generalized signal) characterized by amplitude $|S_0|$ and phase $\angle S_0$, one can always retrieve a pure sinusoid by evaluating the real part

$$\mathtt{Real}(ilde{s}(t)) = \mathtt{Real}(S_0 e^{j\omega t}) = |S_0| \cos(\omega t + \angle S_0)$$

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So, let's consider a 2nd order mechanical system driven by generalized sinusoidal force $ilde f(t)\equiv F_0e^{j\omega t}$ and look for generalized sinusoidal motions $ilde x(t)\equiv X_0e^{j\omega t}$

$$ilde{f}(t) \equiv F_0 e^{j\omega t} \Rightarrow \boxed{ exttt{system}} \Rightarrow ilde{x}(t) \equiv X_0 e^{j\omega t}$$

Note that taking derivatives of generalized sinusoids (i.e. complex exponentials) is particularly computationally straightforward, i.e.

$$rac{d}{dt} ilde{x}(t) = rac{d}{dt}X_0e^{j\omega t} = X_0rac{d}{dt}e^{j\omega t} = j\omega X_0e^{j\omega t} = j\omega ilde{x}(t)$$

so formally $\frac{d}{dt}$ can be replace by a $j\omega$ whenever dealing with generalized sinusoids. Therefore, Newton's law for generalized sinusoids simply becomes

$$ilde{f}(t) = j\omega m(j\omega ilde{x}(t)) + j\omega b ilde{x}(t) + k ilde{x}(t)$$

which, recalling that $j^2 = -1$, becomes

$$ilde{f}(t) = (k - m\omega^2 + j\omega b) ilde{x}(t)$$

Recalling that $ilde{f}(t)=F_0e^{j\omega t}$ and $ilde{x}(t)=X_0e^{j\omega t}$, one gets

$$F_0$$
e $^{j\omega t}=(k-m\omega^2+j\omega b)X_0$ e $^{j\omega t}$

Frequency Response

Frequency response $H(j\omega)$ is a useful concept for linear systems and is usually defined as 'output' over 'input' ratio, which for a spring-mass-damper systems becomes:

$$H(j\omega) := rac{\mathtt{output}}{\mathtt{input}} = rac{ ilde{x}(t)}{ ilde{f}(t)} = rac{X_0 e^{j\omega t}}{F_0 e^{j\omega t}} = rac{1}{k - \omega^2 m + j\omega b}$$

A specific mechanical system is defined when values for $\{m,k,b\}$ are specified. So that $H(j\omega)$ is just a complex function of the independent variable ω .

The same system can be rewritten as

$$H(j\omega) = rac{1}{k-\omega^2 m + j\omega b} = rac{k^{-1}}{1-rac{\omega^2}{\omega_0^2} + jrac{\omega}{Q\omega_0}}$$

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where we conveniently define

- $\omega_0 := \sqrt{k/m}$, also known as **resonance frequency**, merasured in <code>[rad/sec]</code>

• $Q=\sqrt{mk/b^2}$, also know as **quality factor**, unitless

to students:

- 1. verify that with this definition of ω_0 and $Q, H(j\omega)$ can be rewritten as above boxed equation
- 2. verify that Q is unitless

When $\omega=\omega_0$ the real part of the denominator of $H(j\omega)$ becomes zero and one has

$$H(j\omega_0)=-jk^{-1}Q$$

Note: although not the analytical one, ω_0 is a good approximation for the point of maximum of |H|:

- maximizing $\lvert H \rvert^2$ is equivalent to minimizing its inverse

$$1/|H|^2 = (1 - \omega^2/\omega_0^2)^2 + Q^{-2}\omega^2/\omega_0^2$$

- which, $\omega \simeq \omega_0$, can be approximated by $(1-\omega^2/\omega_0^2)^2 + Q^{-2}$,
- this approximation has a minimum at $\omega=\omega_0$

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