IVT

$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

FVT

$(e^{-\infty} = 0)$

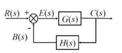
$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$\lim_{t\to\infty} f'(t) = \lim_{s\to 0} s[sX(s) - x(0)]$

☆ Use L'Hospital's rule if $\frac{0}{0}$

eg: $\lim_{s\to\infty} \frac{s}{(s+2)^2}$

Transfer function



Open-Loop TF: $\frac{B(s)}{E(s)} = G(s)H(s)$ Forward Loop TF: $\frac{C(s)}{E(s)} = G(s)$

For negative (positive) feedback system, For unity feedback system,

$\text{CLTF} = \frac{\text{Forward loop TF}}{1 \pm \text{Open loop TF}}$ $\text{CLTF} = \frac{\text{Forward loop TF}}{1 + \text{Forward loop TF}}$

Characteristic eqn (CE) → Roots of this eqn = Poles

Laplace

$\mathcal{L}[f(t)] = F(s)$

$\mathcal{L}[\dot f(t)] = sF(s) - f(0^+)$

$\mathcal{L}^{-1}[F(s)] = f(t)$

Complex numbers

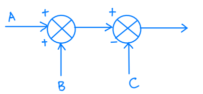
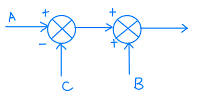
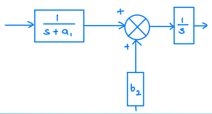
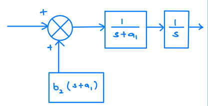
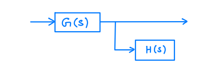
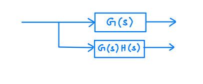
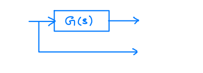
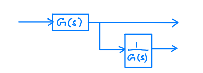
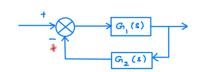
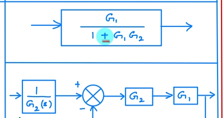
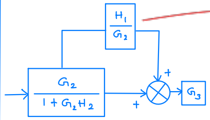
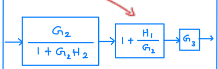
$G(s) = \mathrm{Re}(G(s)) + \mathrm{Im}(G(s))$

$|G(s)| = \sqrt{\mathrm{Re}(G(s))^2 + \mathrm{Im}(G(s))^2}$

$\angle G(s) = \tan^{-1}\frac{\mathrm{Im}(G(s))}{\mathrm{Re}(G(s))}$

1. Sub $s = j\omega$
2. Rationalise w conjugate to make denominator real
3. Split real & imaginary parts

Block diagram

	BEFORE	AFTER
Consecutive summation blocks are swappable		
Moving summation point ahead of a block:		
Moving branch point ahead of a block:		
Moving branch point behind a block:		
Eliminating a feedback loop: [Don't swap branch & ⊗ ×]		 Unity feedback system
Combining feedforward term with unity transfer function		

Partial fractions

$\frac{1}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$

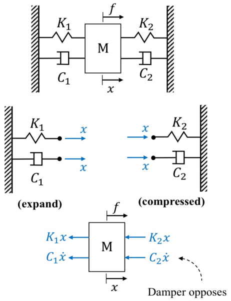
$\frac{1}{s(s+1)(s+10)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+10}$

$\frac{1}{(s-2)(s^2+2s+1)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+2s+1}$

$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$

Forces and displacements



Newton's second law:

$\sum F = m\ddot x$

$f - K_1x - K_2x - C_1\dot x - C_2\dot x = m\ddot x$

$m\ddot x + (C_1 + C_2)\dot x + (K_1 + K_2)x = f$

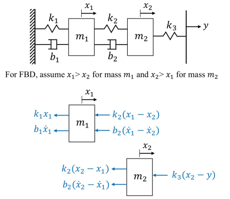
Taking Laplace transform and assuming zero initial condition:

$$\boxed{\begin{aligned}\mathcal{L}\{\ddot x(t)\} &= s^2X(s) - sx(0) - \dot x(0)\\ \mathcal{L}\{\dot x(t)\} &= sX(s) - x(0)\end{aligned}}$$

$m(s^2X(s) - \cancel{sx(0)} - \cancel{\dot x(0)}) + (C_1 + C_2)(sX(s) - \cancel{x(0)}) + (K_1 + K_2)X(s) = F(s)$

$[ms^2 + (C_1 + C_2)s + (K_1 + K_2)]X(s) = F(s)$ always place all X(s) term one side!

$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + (C_1 + C_2)s + (K_1 + K_2)}$



For FBD, assume $x_1 > x_2$ for mass m_1 and $x_2 > x_1$ for mass m_2

Newton's second law for m_1:

$\sum F = m_1\ddot x_1$

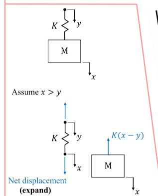
$-k_1x_1 - k_2(x_1 - x_2) - b_1\dot x_1 - b_2(\dot x_1 - \dot x_2) = m_1\ddot x_1$

$m_1\ddot x_1 + (b_1 + b_2)\dot x_1 + (k_1 + k_2)x_1 = b_2\dot x_2 + k_2x_2$

Newton's second law for m_2:

$-k_2(x_2 - x_1) - b_2(\dot x_2 - \dot x_1) - k_3(x_2 - y) = m_2\ddot x_2$

$m_2\ddot x_2 + b_2\dot x_2 + (k_2 + k_3)x_2 = b_2\dot x_1 + k_2x_1 + k_3y$



Assume $x > y$

Newton's second law:

$\sum F = m\ddot x$

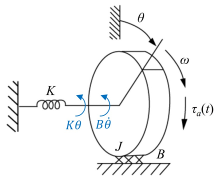
$-K(x - y) = m\ddot x$

$m\ddot x + Kx = Ky$

Laplace transform (assuming zero initial condition):

$[ms^2 + K]X(s) = KY(s)$

$\frac{X(s)}{Y(s)} = \frac{K}{ms^2 + K}$



Note:

Torque τ_a is the input.

Angular displacement θ is the output.

Newton's second law:

$\sum T = J\ddot\theta$

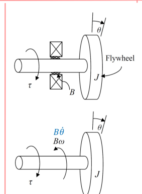
$\tau_a - B\dot\theta - K\theta = J\ddot\theta$

$J\ddot\theta + B\dot\theta + K\theta = \tau_a$

Taking Laplace transform and zero initial condition:

$Js^2\Theta(s) + Bs\,\Theta(s) + K\Theta(s) = T(s)$

$\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K}$



Newton's second law:

$J\ddot\theta + B\dot\theta + \cancel{K\theta} = \tau$

$J\ddot\theta + B\dot\theta = \tau$

$J\dot\omega + B\omega = \tau$

Laplace transform:

$Js\Omega(s) + B\Omega(s) = T(s)$

$\frac{\Omega(s)}{T(s)} = \frac{1}{Js + B} = \frac{1}{100s + 10}$

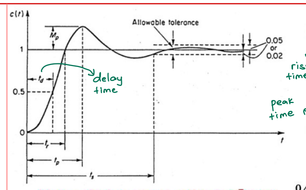
Steady state value: $t \to \infty$

Steps:

1. Define +ve direction
2. Define displacements from rest/equilibrium position for each element in the +ve direction
3. Assume relative displacements across every spring & damper —> visualise if it is in compression or elongation
4. Determine all forces in spring and damper. Forces on inertial elements will have opposite directions
5. For each inertial element, use N2L to determine resultant force producing an acceleration in the +ve direction
6. Laplace transform the ODE to algebra

$f(t)$	$F(s)$
$\delta(t)$ (Impulse fn → Vertical line on graph → ∞)	1
$u_s(t)$ $10u(t)$	$\frac{1}{s}$ $\frac{10}{s}$
t $10t$	$\frac{1}{s^2}$ $\frac{10}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at} $e^{-at}\sin\omega t$	$\frac{1}{(s+a)}$ $\frac{\omega}{(s+a)^2+\omega^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}$	$\frac{1}{(s+a)^n}$
$1 - e^{-at}$	$\frac{a}{s(s+a)}$
$e^{-at} - e^{-bt}$	$\frac{b-a}{(s+a)(s+b)}$
$be^{-bt} - ae^{-at}$	$\frac{(b-a)s}{(s+a)(s+b)}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$e^{-at}\cos bt$	$\frac{s+a}{(s+a)^2+b^2}$
$e^{-at}\sin bt$	$\frac{b}{(s+a)^2+b^2}$
$1 - e^{-at}(\cos bt + \frac{a}{b}\sin bt)$	$\frac{a^2+b^2}{s[(s+a)^2+b^2]}$

2nd order system with $0 < \zeta < 1$



$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi - \beta}{\omega_d} = \frac{\pi - \cos^{-1}\zeta}{\omega_d}$ rise time

$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$ peak time; ω_d: damped natural frequency

Settling time: $t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n}$ (2% criterion)

$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n}$ (5% criterion)

$T = \frac{1}{\zeta\omega_n}$ Time response

$\zeta = \cos\beta$ $\zeta = \frac{-\ln(M_p)}{\sqrt{\pi^2 + \ln^2(M_p)}}$

$\%M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$ Max overshoot

$\cos\beta = \zeta$

DC Gain

$G(s) = \frac{C(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ damping ratio; undamped natural frequency (rad/s)

$\frac{C(s)}{R(s)} = \frac{K}{Ts+1} = \frac{K/T}{s+1/T}$ $K=1$ → Step response

$\zeta < 0.4$ ⇒ Excessive Overshoot

$\zeta > 0.8$ ⇒ Sluggish Response

Damping:

- $0 \le \zeta < 1$ (underdamped)
- $\zeta = 1$ (critically damped)
- $\zeta > 1$ (overdamped)
- $\zeta = 0$ (undamped)

$e_{ss} = r - c_{ss}$

$r = 1$ for unit step input

$= \lim_{s\to 0} sE(s)$

$c(t) = (1 - e^{-\frac{t}{T}}) = u(t)$? response to step input in time domain

$\frac{d}{dt}c(t) = \frac{1}{T}e^{-\frac{t}{T}}$ response to impulse input in time domain

Transient Poles

$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

$Ae^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}\,t + \phi)$

Eg: $R(s) = \frac{1}{s}$ → Unit step input

$\Rightarrow c_{RSS} = \lim_{s\to 0} sC(s) = \lim_{s\to 0} s\frac{C(s)}{R(s)}R(s) = \lim_{s\to 0} s\frac{0.1K}{s^2 + 6s + 0.1K}\cdot\frac{1}{s} = 1$ STEADY STATE RESPONSE!

SCALING INPUT

If input is not unit Eg: Step input Mag of 10 instead of 1 Change to $\frac{10}{1+K_p}$

Summary

→ for stable systems & unity feedback only

Input "level"

System Types	Steady-state Error (e_{ss})		
	Unit Step input	Unit Ramp input	Unit Parabolic input
Type 0	$\frac{1}{1+K_p}$	(too slow to follow input) ∞ ∴ ∞ steady state errors)	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$
Type 3	0	0	0

System "level"

Input "Level" higher than System "level" $e_{ss} \to \infty$

Input "Level" equal to System "level" e_{ss} is non-zero

Input "Level" lower than System "level" $e_{ss} = 0$

DC Gain: $\lim_{s\to 0} G(s) = \lim_{s\to 0}\frac{C(s)}{R(s)}$

DC Gain for a step input = Steady state response

∞: system too slow to follow input

0: sufficient integrators fully eliminate e_{ss}

Type N: N number of integrators

$\frac{1}{T} = \frac{1}{\pi}$ $\omega = 2\pi f$ TF

$E(s) = R(s) - C(s) \to \frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)}$

Static Position error constant, $K_p = \lim_{s\to 0} G(s)$

Static Velocity error constant, $K_v = \lim_{s\to 0} sG(s)$

Static Acceleration error constant, $K_a = \lim_{s\to 0} s^2G(s)$

Forward / Open loop for unity feedback

General Formula: $e_{ss} = \lim_{s\to 0}\frac{sR}{(1+G)} = \frac{1}{1+K_p} = \frac{1}{K_v}$

$G(s) = \frac{K(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^N(T_1 s+1)(T_2 s+2)\cdots(T_p s+1)}$ → Type N System

↳ N is a trade off between steady-state error and stability.

↳ Increase N, decrease steady-state error (desirable) and decrease stability (undesirable)

Increasing system type requires an integrator ($\frac{1}{s}$) in the forward path but this will have destabilizing effect.

Eg: $\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Bs + K} = \frac{\frac{K}{J}}{s^2 + \frac{B}{J}s + \frac{K}{J}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \to \omega_n^2 = \frac{K}{J} \Rightarrow \omega_n \propto \sqrt{K}$

$2\zeta\omega_n = \frac{B}{J} \Rightarrow \zeta = \frac{B}{2J\omega_n} \Rightarrow \zeta \propto \frac{1}{\sqrt K}$ ↓ → More oscillatory

$\Rightarrow \zeta \propto B$

☆ Eg: $\lim_{s\to 0}\frac{5}{s + sG(s)} = \frac{5}{\lim_{s\to 0} sG(s)}$

Dominant Poles

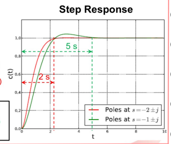
left: fast right: slow

Dominant poles are the rightmost poles (slowest)

e.g., compare step response of two systems

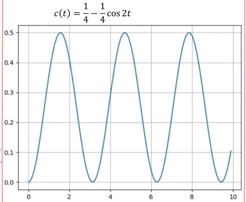
System #1 has two poles: $s = -2 \pm j$ transient response: $Ae^{-2t}\cos(t + \phi)$

System #2 has two poles: $s = -1 \pm j$ transient response: $Ae^{-t}\cos(t + \phi)$ Slower!



$C(s) = \frac{1}{4s} - \frac{1}{4}\frac{s}{(s^2+4)}$

$c(t) = \frac{1}{4} - \frac{1}{4}\cos 2t$



Properties \ Controllers	K_p (Proportional)	$K_d s$ (Derivative)	K_i/s (Integral)
Stability	↓	↑	↓
Steady-State Error (e_{ss})	↑	—	↑
Speed	↑	↑	↑
Disturbance Rejection	↑	—	↑
Noise Rejection	—	↓	↑

↑ Good — Neutral ↓ Bad

PI Controller

I: Improves steady state response

P: Improves transient response (faster response)

∴ PI: Improves both

PD Controller

PD controller is able to anticipate error to improves stability, i.e., to increase the damping of the system.

It tends to amplify high frequency signals, which may lead to saturation of system. → PD not suitable for system with high freq. input

D controller is always combined with a P or PI controller.

Eg: If system oscillating too much due to low damping, use PD controller to minimise oscillation in response.

PIs add DC armature eqn.

Controllers

Proportional control improves speed of response and disturbance rejection.

Integral control can remove steady-state error. However, it may also lead to oscillatory response or instability.

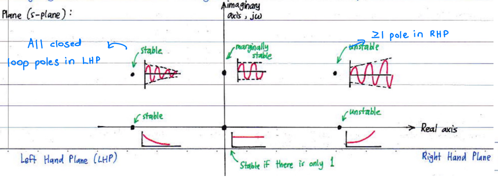
Derivative control adds damping to the system and tends to increase the stability of the system.

Derivative control permits the use of a larger proportional gain, which will result in an improvement in the steady-state accuracy. → limit of stability in system to proportional gain will increase. Hence, more stable. Allows increase in proportional gain in controller

Stability

	Natural Response
System Stable:	Output of system converges to a particular value/state for a constant input.
System Unstable:	Output of system goes to infinity for finite (or constant) input.
System Marginally Stable:	Continuous oscillatory output for a constant input

Complex Plane (s-plane):



Routh-Hurwitz

(Use CE)

s^6	a_0	a_2	a_4	a_6	
s^5	a_1	a_3	a_5	0	
s^4	b_1	b_2	b_3		
s^3	c_1	c_2			
s^2	⋮				
s^1					
s^0					

→ Form first 2 rows using the coefficients of the s [up-down, up-down]

→ Any blanks fill with zero

→ Calculate array until the end

$b_1 = \frac{a_1a_2 - a_0a_3}{a_1}$, $b_2 = \frac{a_1a_4 - a_0a_5}{a_1}$, …

$c_1 = \frac{b_1a_3 - a_1b_2}{b_1}$, $c_2 = \frac{b_1a_5 - a_1b_3}{b_1}$, …

In first column,

↳ No changing of sign → system stable

↳ Sign changes (E.g. +ve to −ve) → Number of times the sign changes represents the number of poles in RHP

Form column with the highest power of s

Steady-state response → System accuracy

Transient response → System speed

Special case ①

- If 1st element in any row is zero, but others are non-zero replace 0 with ε > 0 and proceed as before
- If sign above zero (ε) is the same as below, it indicates that there are a pair of imaginary roots
- If sign above zero (ε) is opposite to that of below it, it indicates that there is a sign change ⇒ unstable system

Special Case 2

If the entire row of array is zero.

1. Form auxiliary polynomial from coefficients of row immediately above it

↳ If the row above zero row is s^n where n is even

E.g. s^4 a_1 a_2 a_3 → $P(s) = a_1s^4 + a_2s^2 + a_3s^0$

s^3 0 0

↳ $n \ge 0$, each subsequent power −2 the previous one

↳ If the row above zero row is s^n where n is odd

E.g. s^5 a_1 a_2 a_3 a_4 → $P(s) = a_1s^5 + a_2s^3 + a_3s^1$

s^4 0 0 0 0

↳ $n \ge 0$, each subsequent power −2 the previous one

2. Differentiate the equation and replace the 0s with the coefficients of the differentiated value.
3. Continue as per normal for the array.

Root Locus → ☆ MUST BE SYMMETRICAL ABOUT REAL AXIS

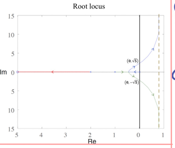
☆ If poles only move along imaginary axis, K cannot be tuned to obtain a stable system

1. Starting and ending points of locus
   ① Find CE (From OLTF) — poles (X)
   ② Sub K=0 and find starting points (n)
   Sub K=∞ and find ending points (m)
   Finite GH zeros (O) — Infinite zeros — End directions given by asymptotes
   If no finite zeros, ending is @ infinity

$1 + K\frac{(1+0.2s)}{s(s+1)(s+2)} = 0$ — If OLTF $OLTF = K\frac{N(s)}{D(s)}$

Starting points: $s = 0, s = -1, s = -2$ (x)
Ending points: $s = -5$ (finite zero) and two infinite zeros (O)

☆ For K to be ∞, RHS should divide by zero



2. Locus on real axis
   Segments on the real axis to the left of odd number real-axis open-loop poles and zeros are parts of root locus.
   One real axis segment between −2 and −5, and the other segment between 0 and −1 will break out somewhere between them.

3. Break-in and break-out points
   ① Make K the subject of CE (from CLTF)
   ② $\frac{dK}{ds} = 0$ and find break-in/out points — minimum K, max K
   ☆ If point is not on locus, reject it

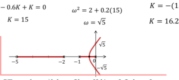
$K = \frac{-s(s+1)(s+2)}{(1+0.2s)}$
$\frac{dK}{ds} = -\frac{(1+0.2s)(3s^2+6s+2) - s(s+1)(s+2)(0.2)}{(1+0.2s)^2} = 0$
$(1+0.2s)(3s^2+6s+2) - s(s+1)(s+2)(0.2) = 0$
$(5+s)(3s^2+6s+2) - s(s+1)(s+2) = 0$
$(3s^3+21s^2+32s+10) - (s^3+3s^2+2s) = 0$
$2s^3+18s^2+30s+10 = 0$
$s = -6.934, -1.6091, -0.4475$ (rejected) (rejected)

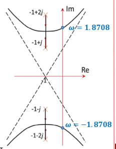
$K = -(s^2+2s+2)(s^2+2s+5)$
$\frac{dK}{ds} = -(2s+2)(s^2+2s+5) - (2s+2)(s^2+2s+2) = 0$
$2(s+1)(2s^2+4s+7) = 0$
$s = -1$ or $-1 \pm j1.58$ — In between the open loop poles
not on locus & $K = -4$ (rejected) → Break-out points

4. Imaginary axis intersection
   ① Sub $s = j\omega$ into CE (From CLTF)
   ② Separate real & imaginary parts
   Find Gain, K ← Sub into ← Find ω → lies on imaginary axis

Set $s = j\omega$ in CE: $s(s+1)(s+2) + K(1+0.2s) = 0$
$j\omega(j\omega+1)(j\omega+2) + K(1+0.2j\omega) = 0$
$(-3\omega^2+K) + j(-\omega^3+(2+0.2K)\omega) = 0$
$-3\omega^2+K = 0$; $-\omega^3+(2+0.2K)\omega = 0$
$-3(2+0.2K)+K = 0$; $\omega^2 = 2+0.2K$
$-6 - 0.6K + K = 0$; $\omega^2 = 2+0.2(15)$
$K = 15$; $\omega = \sqrt{5}$

CE: $1 + \frac{K}{(s^2+2s+2)(s^2+2s+5)} = 0$
$(s^2+2s+2)(s^2+2s+5) + K = 0$
$((j\omega)^2+2(j\omega)+2)((j\omega)^2+2(j\omega)+5) + K = 0$
$(-\omega^2+2j\omega+2)(-\omega^2+2j\omega+5) + K = 0$
$\omega^4 - 11\omega^2 + 10 + K = 0$; $-4\omega^3 + 14\omega = 0$; $\omega = \pm 1.8708$
$K = -(1.8708)^4 + 11(1.8708)^2 - 10$
$K = 16.2495$





5. Asymptotes (infinite open-loop zeros)
   Angles: Asymptote angle: $\angle s = \frac{\pi(1+2q)}{[n-m]}$, $q = 0, \ldots, n-m-1$ (no. of zeros, no. of poles)
   $\angle s = \pm \ldots$
   Location: Real axis intercept: $\sigma_a = \frac{\sum_1^n(-p_i) - \sum_1^m(-z_i)}{n-m} = \frac{\sum GH \text{ finite poles} - \sum GH \text{ finite zeros}}{n-m}$
   ☆ Note that imaginary parts will cancel off

CE: $s(s+1)(s+2) + K(1+0.2s) = 0$
Angles:
$\angle s = \frac{\pi(1+2q)}{[n-m]}$, $q = 0, \ldots, n-m-1$
$= \frac{\pi(1+2q)}{[3-1]}$, $q = 0, 1$
$= \frac{\pi}{2}, \frac{3\pi}{2}$

$\angle s = \frac{\pi(1+2q)}{4-0} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$, $q = 0, \ldots, 3$ (±45°, ±135°)

Real axis intercept:
$\sigma_a = \frac{\sum GH \text{ finite poles} - \sum GH \text{ finite zeros}}{n-m}$
$= \frac{0-1-2-(-5)}{2} = 1$
$\sigma_a = \frac{\sum_1^n(-1+j)+(-1-j)+(-1+2j)+(-1-2j)}{4} = \frac{-4}{4} = -1$

This means that part of the locus will cross over the RHP

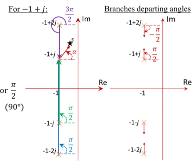
6. Departure and arrival angles
   ↳ Only for complex poles/zeros ☆

Departure angle: $\sum_{i=1}^m \angle(s+z_i) - \sum_{j=1}^n \angle(s+p_j) = -\pi$

Arrival angle: $\sum_{i=1}^m \angle(s+z_i) - \sum_{j=1}^n \angle(s+p_j) = \pi$

For $-1+j$: (no finite zeros) — Take note of the approximation in the angles
$\Rightarrow 0 - \sum_{j=1}^{n=4} \angle(s+p_j) = -(\alpha + \frac{\pi}{2} + \frac{\pi}{2} + \frac{3\pi}{2}) = -\pi$ ⇒ $\alpha = -\frac{3\pi}{2}$ or $\frac{\pi}{2}$ (90°)

For $-1+2j$:
$\Rightarrow 0 - \sum_{j=1}^{n=4} \angle(s+p_j) = -(\alpha + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}) = -\pi$ ⇒ $\alpha = -\frac{\pi}{2}$ (−90°)



Controller design

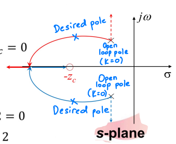
① PD Controller $G_c = K_p + K_D s$
closed-loop poles changed
$= K_D\left(s + \frac{K_p}{K_D}\right)$
Transient response improved
$= K_D(s + z_c)$
1) Determine desired closed loop poles from desired transient response
2) Express CE of CLTF in terms of K_p and K_D
3) Substitute desired poles into CE
4) Solve the real and imaginary parts of CE to find K_p and K_D
5) TF for controller can be found
Drawbacks: ① cannot be realised with passive components ② sensitive to high frequency noise

② Lead compensator
$G_c = K\frac{(s+z_c)}{(s+p_c)}$; $z_c < p_c$
1) Design the desired transient response with a PD controller ∴ $G_c = K_D(s+z_c')$
2) Using the obtained PD controller, specify a value of z_c such that $z_c < z_c'$
3) Use the CE of CLTF to determine K and p_c by substituting desired poles

① PD Controller: $G_c(s) = K(s+z_c)$ — K_D and z_c change closed-loop poles

CLTF: $\frac{C(s)}{R(s)} = \frac{K(s+z_c)(\frac{1}{s^2})}{1+K(s+z_c)(\frac{1}{s^2})} = \frac{K(s+z_c)}{s^2+Ks+Kz_c}$

CE: $s^2 + Ks + Kz_c = 0$
Subst $s = -1+j$: $(-1+j)^2 + K(-1+j) + Kz_c = 0$
$-2j - K + Kj + Kz_c = 0$
$-K + Kz_c + j(K-2) = 0$
$\Rightarrow -K + Kz_c = 0$; $\Rightarrow K - 2 = 0$
$z_c = 1$; $K = 2$

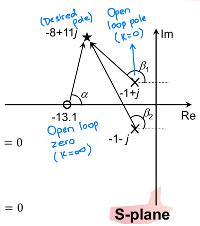


Changing poles to increase stability and speed up settling time with increased damping

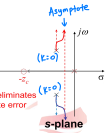
② Lead Compensator: $G_c(s) = K\frac{(s+z_c)}{(s+p_c)}$, $p_c > z_c$
Choose: $z_c \le 1$ (z_c of PD controller) ⇒ $z_c = 0.5$

$\frac{C(s)}{R(s)} = \frac{K\frac{(s+z_c)}{(s+p_c)}(\frac{1}{s^2})}{1+K\frac{(s+z_c)}{(s+p_c)}(\frac{1}{s^2})} = \frac{K(s+z_c)}{s^2(s+p_c)+K(s+z_c)}$

Subst $z_c = 0.5$ & $s = -1+j$:
CE: $s^2(s+p_c) + K(s+z_c) = 0$
$(-1+j)^2(-1+j+p_c) + K(-1+j+0.5) = 0$
$-2j(-1+j+p_c) + K(0.5+j) = 0$
$2 - 0.5K + j(-2p_c + K + 2) = 0$
$\Rightarrow 2 - 0.5K = 0$; $\Rightarrow -2p_c + K + 2 = 0$
$K = 4$; $p_c = 3$



③ PI Controller → eliminates $e_{ss}(t)$
$G_c = K_p + \frac{K_I}{s} = \frac{K_p\left(s + \frac{K_I}{K_p}\right)}{s} = \frac{K_p(s+z_c)}{s}$
If z_c is small ~ pole-zero cancellation (transient response slightly affected)
Design steps similar to PD controller
Drawback: ① cannot be realised with passive components



PI controller eliminates steady-state error

④ Lag compensator
$G_c = K\frac{(s+z_c)}{(s+p_c)}$; $p_c \ll z_c < 1$ ~ pole-zero cancellation

Lag Compensator: $G_c(s) = K\frac{(s+z_c)}{(s+p_c)}$, $p_c \ll z_c \ll 1$

$G(s) = K\frac{(s+z_c)}{(s+p_c)}\frac{1}{s(s+10)(s+20)}$

⑤ PID Controller
$G_c = K_p + K_D s + \frac{K_I}{s} = \frac{K_D(s+z_1)(s+z_2)}{s}$ — PD, PI
1) Use PD controller to tune transient response
2) Use PI controller to eliminate steady-state error

$\Rightarrow e_{ss} = \frac{1}{K_v} \times 2\pi = \frac{1}{\lim_{s\to0} sG(s)} \times 2\pi = \frac{1}{820\frac{(z_c)}{(p_c)}\frac{1}{(10)(20)}} \times 2\pi = 1.53\frac{p_c}{z_c}$ rad

⑥ Lead-lag compensator
$G_c = \frac{K_1(s+z_1)}{(s+p_1)} \cdot \frac{K_2(s+z_2)}{(s+p_2)}$
1) Design the lead compensator to tune transient response
2) Design the lag compensator to reduce steady-state error

Open loop system still type 1, ramp input
$e_{ss} = 1.53\frac{p_c}{z_c}$ rad ⇒ For the new e_{ss} to be $\frac{1}{10}$ of the previous one, $\frac{p_c}{z_c} = \frac{1}{10}$
Choose e.g. $p_c = 0.05$ & $z_c = 0.5$

Open Loop TF for Root Locus (with K):
Previous: $G(s) = \frac{K}{s(s+10)(s+20)}$
New: $G(s) = K\frac{(s+0.5)}{(s+0.05)}\frac{1}{s(s+10)(s+20)}$
When K = 820, closed-loop poles: (get CLTF)
Previous: $s = -3.6 \pm 4.8j$
New: $s = -3.36 \pm 4.54j$ ← CE
Root locus does not change much!
Reduces e_{ss}, small change in transient response

DESIRED Poles: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \approx -8 \pm 11j$

Bode Plot Sketching

Input: $r = r_0\cos(\omega t)$ ~ harmonic input
Steady-state: $c_{ss} = [r_0 |G(s=j\omega)|]\cos(\omega t + \angle G(s=j\omega))$ — factor of magnitude change; phase shift

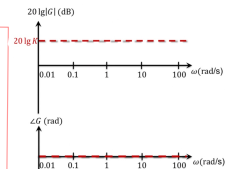
Eg: Transfer function → $\frac{2}{s+1}$
Input → $r(t) = r_0\cos(10t)$, $\omega = 10$
Sub $s = 10j$: $G(s=10j) = \frac{2}{10j+1} = \frac{2}{\sqrt{104}\,e^{j(\tan^{-1}5)}} = \frac{2}{\sqrt{104}}e^{-j(\tan^{-1}5)}$
By comparison, $c_{ss} = |G(s=j\omega)| = \frac{2}{\sqrt{104}}$ & $\angle G(s=j\omega) = -\tan^{-1}5$

Frequency Response
$G(s=j\omega) = \frac{K\prod_{i=1}^m(j\omega+z_i)}{\prod_{j=1}^n(j\omega+p_j)}$

$R(s) \to \boxed{G(s)} \to C(s)$ Transfer Function

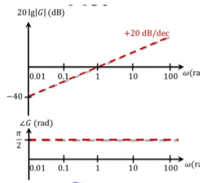
① Linear combination of angles
$\angle G(s=j\omega) = \sum_{i=1}^m \tan^{-1}\left(\frac{\omega}{z_i}\right) - \sum_{j=1}^n \tan^{-1}\left(\frac{\omega}{p_j}\right)$
② Linear combination of magnitudes
$20\lg|G(s=j\omega)| = 20\lg(K) + \sum_{i=1}^m 20\lg\left(\sqrt{\omega^2+z_i^2}\right) - \sum_{j=1}^n 20\lg\sqrt{\omega^2+p_j^2}$

Principal components
① Constant



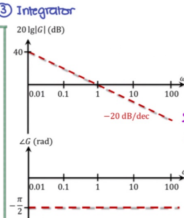
$G(s) = K$
$G(s=j\omega) = Ke^{j0}$
$20\lg|G(s=j\omega)| = 20\lg K$
$\angle G(s=j\omega) = 0$

② Differentiator ☆ always 'rmb' break down eqn ↳ $\left(\frac{1}{2}\right)\left(\frac{a}{s+a}\right)\ldots$



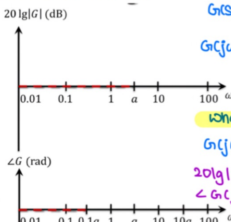
$G(s) = s$
$G(s=j\omega) = j\omega = \omega e^{j\frac{\pi}{2}}$
$20\lg|G(s=j\omega)| = 20\lg\omega$
$\angle G(s=j\omega) = \frac{\pi}{2}$

③ Integrator



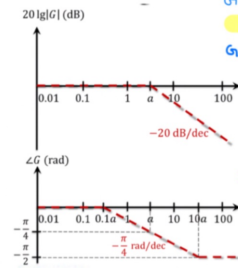
$G(s) = \frac{1}{s}$
$G(s=j\omega) = \frac{1}{j\omega} = \frac{1}{\omega}e^{-j\frac{\pi}{2}}$
$20\lg|G(j\omega)| = -20\lg\omega$
$\angle G(j\omega) = -\frac{\pi}{2}$

④ First Order Pole (I)



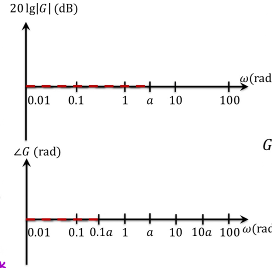
$G(s) = \frac{a}{s+a}$
$G(j\omega) = \frac{a}{j\omega+a}$
when $\omega \ll a$
$G(j\omega) = 1 = 1e^{j0}$
$20\lg|G(j\omega)| = 0$
$\angle G(j\omega) = 0$

First order Pole (II)



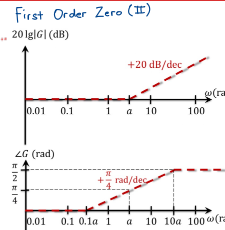
$G(s=j\omega) = \frac{a}{j\omega+a}$
when $\omega \gg a$
$G(j\omega) = \frac{a}{j\omega}$
$= \frac{a}{\omega}e^{j(-\frac{\pi}{2})}$
$20\lg|G(j\omega)| = 20\lg\frac{a}{\omega}$
$\angle G(j\omega) = -\frac{\pi}{2}$
Corner f: for mag, $\omega = a$; for phase, $\omega = 0.1a, 10a$ (at the corner it turns)

⑤ First Order Zero (I)



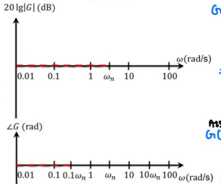
$R(s) \to \boxed{\frac{s+a}{a}} \to C(s)$ Transfer Function
$G(s=j\omega) = \frac{j\omega+a}{a}$
When $\omega \ll a$
$G(s=j\omega) = \frac{a}{a} = 1e^{j0}$
Magnitude
$20\lg|G(j\omega)| = 0$
Phase
$\angle G(j\omega) = 0$

First Order Zero (II)



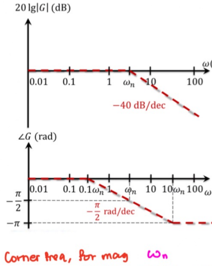
$R(s) \to \boxed{\frac{s+a}{a}} \to C(s)$ Transfer Function
$G(s=j\omega) = \frac{j\omega+a}{a}$
When $\omega \gg a$
$G(s=j\omega) = \frac{j\omega}{a} = \frac{\omega}{a}e^{j(\frac{\pi}{2})}$
Magnitude
$20\lg|G(j\omega)| = 20\lg\frac{\omega}{a} = 20\lg\omega - 20\lg a$
Phase
$\angle G(j\omega) = \frac{\pi}{2}$
Corner f: for mag, $\omega = a$; for phase, $\omega = 0.1a, 10a$ (at the corner it turns)

⑥ Second order Pole (I)



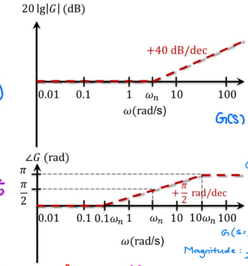
(K=1)
$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
$G(s=j\omega) = \frac{\omega_n^2}{(-\omega)^2+2\zeta\omega_n\omega+\omega_n^2}$
when $\omega \ll \omega_n$, Assume large ζ
$G(s=j\omega) = \frac{\omega_n^2}{\omega_n^2} = 1e^{j0}$
$20\lg|G(j\omega)| = 0$
$\angle G(j\omega) = 0$

Second Order Pole (II)



when $\omega \gg \omega_n$
$G(s=j\omega) = \frac{\omega_n^2}{-\omega^2} = \frac{\omega_n^2}{\omega^2}e^{j(-\pi)}$
$20\lg|G(j\omega)| = 20\lg\frac{\omega_n^2}{\omega^2} = 40\lg\omega_n - 40\lg\omega$
$\angle G(j\omega) = -\pi$
Corner freq, for mag ω_n; for phase $0.1\omega_n, 10\omega_n$

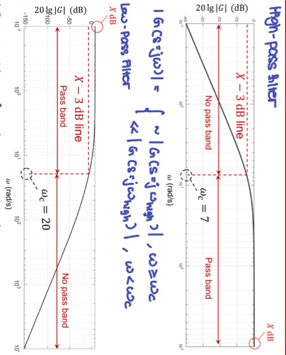
⑦ Second-order zero [Reflection of second order pole along x-axis]



At ω_n, there is a peak for magnitude.
Before ω_n, mag is const
$G(s) = \frac{s^2+2\zeta\omega_n s+\omega_n^2}{\omega_n^2}$
$G(s=j\omega) = \frac{(j\omega)^2+2\zeta j\omega_n\omega+\omega_n^2}{\omega_n^2}$
When $\omega \gg \omega_n$,
$G(s=j\omega) = -\frac{\omega^2}{\omega_n^2} = \frac{\omega^2}{\omega_n^2}e^{j\pi}$
Magnitude: $20\lg|G(s=j\omega)| = 20\lg\frac{\omega^2}{\omega_n^2} = 40\lg\omega - 40\lg\omega_n$
Phase: $\angle G(j\omega) = \pi$
Corner freq, for mag ω_n; for phase $0.1\omega_n, 10\omega_n$
Resonance f, $\omega_R = \omega_n\sqrt{1-2\zeta^2}$ — When $0 < \zeta < 0.707$ resonance occurs
Lower ζ = Higher peak & higher resonance
Note: for second-order systems, smaller ζ means sharper peak at ω_n in mag plot & sharper drop at ω_n in phase plot
Min: Zero (=0) ← At the tip of peak

Types of TF pass filters:

Low-pass:	$H(s) = \frac{K}{s^2+bs+a}$
Bandpass:	$H(s) = \frac{Ks}{s^2+bs+a}$
High-pass:	$H(s) = \frac{Ks^2}{s^2+bs+a}$

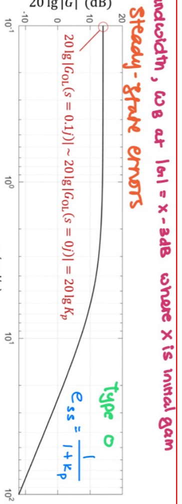


$|G(s=j\omega)| = \begin{cases} \ll |G(s=j\omega)|, & \omega \gg \omega_c \\ \sim |G(s=j\omega_c)|, & \omega \ll \omega_c \end{cases}$ — Low-pass filter

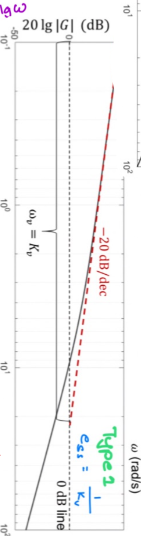
$|G(s=j\omega)| = \begin{cases} \sim |G(s=j\omega_c)|, & \omega \gg \omega_c \\ \ll |G(s=j\omega)|, & \omega \ll \omega_c \end{cases}$ — High-pass filter

Bandwidth, ω_B at $|G| = X - 3$dB where X is initial gain

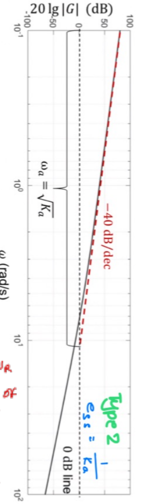
Steady-state errors



Type 0: $e_{ss} = \frac{1}{1+K_p}$

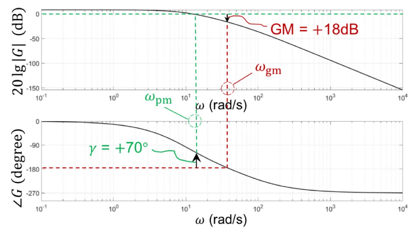


Type 1: $e_{ss} = \frac{1}{K_v}$



Type 2: $e_{ss} = \frac{1}{K_a}$

Bode Plot: Unity feedback
Gain margin GM at $\angle G = -\pi$ is $0 - |G(s=j\omega_{gm})|_{dB}$
Phase margin γ at $|G| = 0$dB is $\angle G(s=j\omega_{pm}) - (-\pi)$



Use bode plot to reverse eng. → get ω_R ← Peak of amp.

Stability:
→ unity feedback is stable if and only if γ > 0 and GM > 0

Closed-loop ζ_{cl}
→ γ ↑ implies ζ_{cl} ↑
Complex funct.
Find c_{ss} using graph
$r(t) = \cos t$, $\omega = 1$ → draw 1 line from phase to mag → get the phase & mag