NANYANG TECHNOLOGICAL UNIVERSITY

# MA2011 MECHATRONICS SYSTEMS INTERFACING

Tutorial 4

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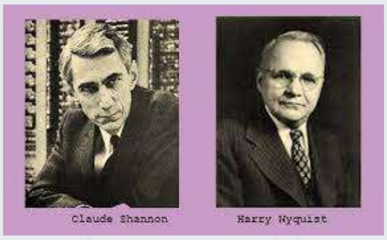
College of Engineering
School of Mechanical and Aerospace Engineering

1

## Sampling




**Shannon-Nyquist Theorem**

- We need to sample a digital signal at a rate more than two times the maximum frequency component in the signal to retain all frequency components.

- In other words, to faithfully represent the analog signal, the digital samples must be taken at a frequency $f_s$, such that

$$f_s > 2f_{max}$$

$f_s$ is called sampling rate, and $f_{max}$ is called Nyquist frequency





2

# Sampling

$$F(t) = C_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} B_n \sin(n\omega_0 t)$$

$$F(t) \simeq C_0 + \sum_{n=1}^{N} A_n \cos(n\omega_0 t) + \sum_{n=1}^{N} B_n \sin(n\omega_0 t)$$

- If we approximate a signal by a truncated Fourier series, the maximum frequency component is the highest harmonic frequency. Then the time interval between the digital samples is

$$\Delta t = 1/\, f_s$$

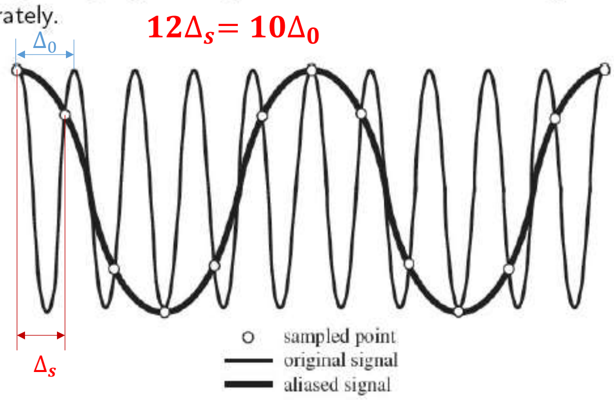
 



# ALIASING

If sampling rate is too low, we obtain wrong digital signal. This is called alaising.
Example: 12 equally spaced samples are taken over 10 cycles of an analog signal, so $\boxed{f_s = 1.2 f_0}$ with $f_0$ frequency of the analog signal. Since $\boxed{f_s < 2f_0,}$ digitized signal does not describe the original signal accurately.

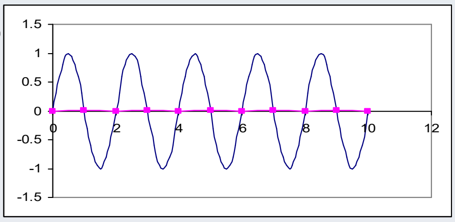
$$12\Delta_s = 10\Delta_0$$



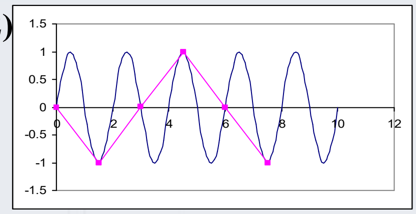

 

## DISCUSSION Q1(A)

1)




2)

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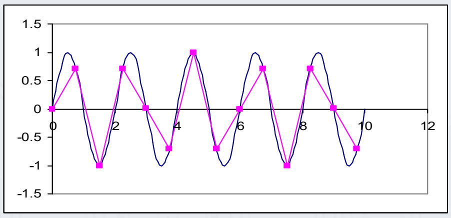
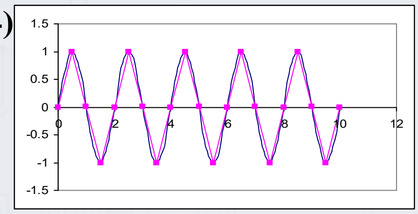
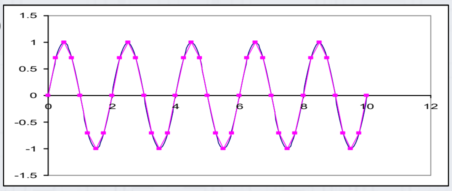



4)




5)




Sampling time interval **Δs** =?

Sampling rate $f_s$=?

Wave Frequency = ?

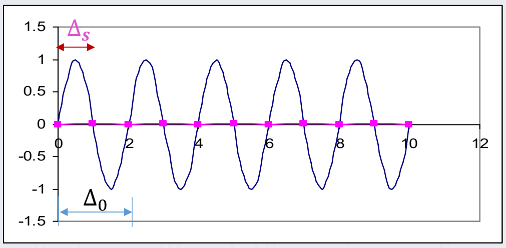
Sampling rate = ? X Wave Frequency






## DISCUSSION Q1(A)

**POOR SAMPLING**




Sampling time interval **Δs** =?

Sampling rate $f_s$=?

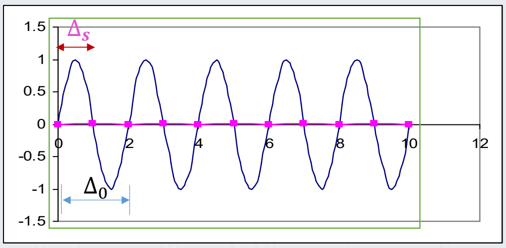
Wave Frequency = ?

Sampling rate = ? X Wave Frequency

## DISCUSSION Q1(A)

**POOR SAMPLING**

$10\Delta_s = 5\Delta_0$ $\quad \frac{f_s}{f_o} = \frac{\Delta_0}{\Delta_s} = 2$




Sampling time interval Δs =1s =(2s/2)

Sampling rate $f_s$=1/ Δs=1 Hz

$\Delta_0$**=2s,** Wave Frequency =1/ $\Delta_0$ =1/2Hz=0.5 Hz

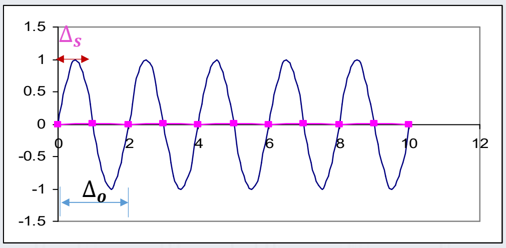
Sampling rate = 2 X Wave Frequency



## DISCUSSION Q1(A)

**POOR SAMPLING**




Wave Frequency = 1/2Hs=0.5 Hz

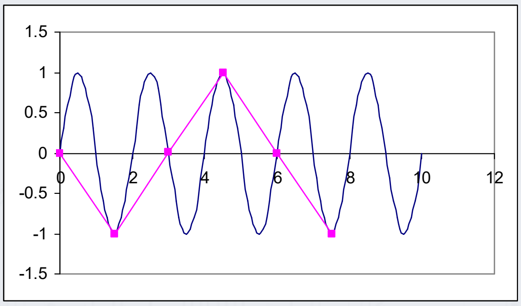
Nyquist Frequency = Wave Frequency=1/2 Hz=0.5 Hz

Sampling rate (=1 Hz) !> 2*Nyquist Frequency (=1Hz)

## DISCUSSION Q1(B)

**POOR SAMPLING**




Sampling time interval **Δs** =?

Sampling rate $f_s$=?

$\Delta_0$**=?** Wave Frequency = ?

Sampling rate = ? X  Wave Frequency





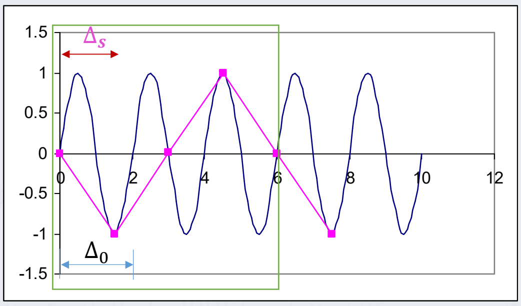


## ANSWER TO Q1(B)

**POOR SAMPLING**

$$4\Delta_s = 3\Delta_0$$

$$\frac{f_s}{f_o} = \frac{\Delta_0}{\Delta_s} = 1.333$$




Sampling time interval Δs =6/4=1.5s

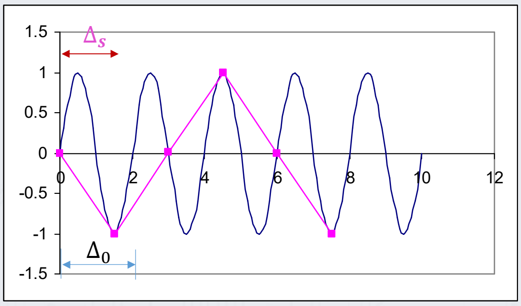
Sampling rate $f_s$=1/ Δs=2/3 Hz=0.666Hz

$\Delta_0$**=2s** , Wave Frequency =1/ $\Delta_0$= 1/2Hz=0.5 Hz

Sampling rate = 4/3 X  Wave Frequency

# DISCUSSION Q1 (B)

**POOR SAMPLING**




Wave Frequency = 1/2s=0.5 Hz

Nyquist Frequency = Wave Frequency=1/2 Hz=0.5Hz

Sampling rate (=2/3 Hz) !> 2*Nyquist Frequency

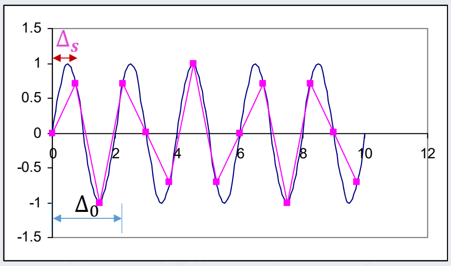






# DISUCSSION Q1(C)

**HIGHER SAMPLING FREQUENCY**




Sampling time interval **Δs** =?

Sampling rate $f_s$=?

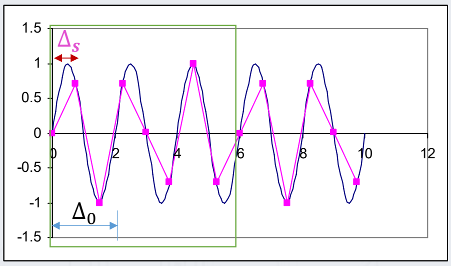
$\Delta_0$**=?** Wave Frequency = ?

Sampling rate = ? X Wave Frequency

# DISCUSSION Q1(C)

**HIGHER SAMPLING FREQUENCY**

$8\Delta_s = 3\Delta_0$ $\qquad \frac{f_s}{f_o} = \frac{\Delta_0}{\Delta_s} = 2.667$




Sampling time interval Δs =6/8s=0.75s

Sampling rate $f_s$=1/ Δs=4/3 Hz=1.333Hz

$\Delta_0$**=2s** , Wave Frequency =1/ $\Delta_0$= 1/2Hz=0.5 Hz

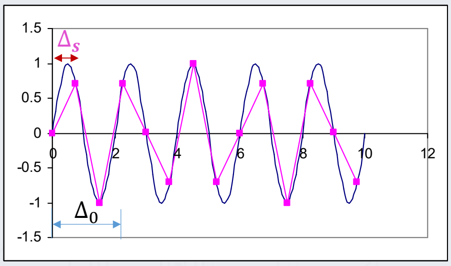
Sampling rate = 8/3 X Wave Frequency



# DISCUSSION Q1(C)

**HIGHER SAMPLING FREQUENCY**




Wave Frequency = 1/2s=0.5 Hz

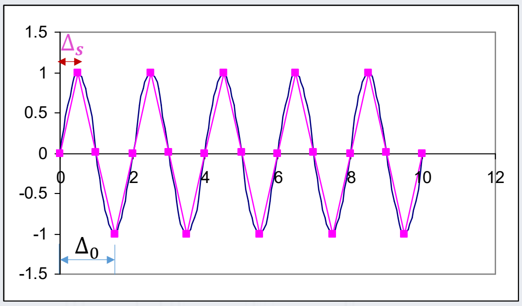
Nyquist Frequency = Wave Frequency=1/2 Hz=0.5Hz

Sampling rate (=4/3 Hz) > 2*Nyquist Frequency (=1Hz)

# DISUCSSION Q1(D)

**GETTING BETTER**




Sampling time interval **Δs** =?

Sampling rate $f_s$=?

$\Delta_0$**=?** Wave Frequency = ?

Sampling rate = ? X  Wave Frequency





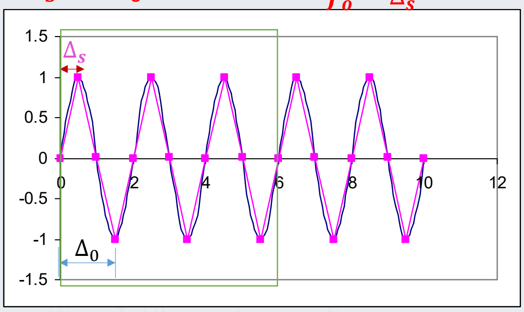


DISUCSSION Q1(D)

**GETTING BETTER**

$12\Delta_s = 3\Delta_0$

$\frac{f_s}{f_o} = \frac{\Delta_0}{\Delta_s} = 4$




Sampling time interval Δs =2/4=0.5s

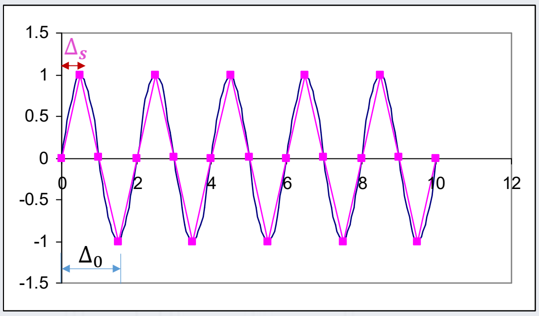
Sampling rate $f_s$=1/ Δt=2 Hz

$\Delta_0$**=2s** , Wave Frequency =1/ $\Delta_0$**=** 1/2Hz=0.5 Hz

Sampling rate = 4 X  Wave Frequency

# DISCUSSION Q1(D)

**GETTING BETTER**




Wave Frequency = 1/2Hz=0.5 Hz

Nyquist Frequency = Wave Frequency=1/2 Hz

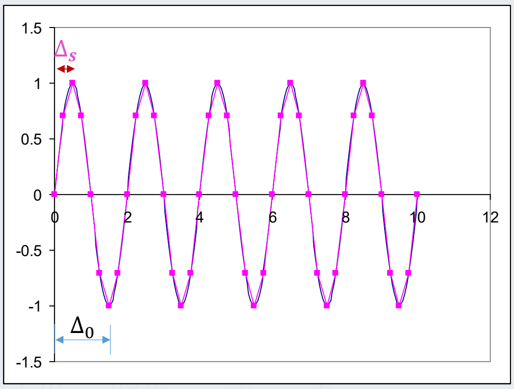
Sampling rate (=2 Hz) > 2*Nyquist Frequency (=1Hz)






# DISUCSSION Q1(E)

**GOOD SAMPLING**




Sampling time interval **Δs** =?

Sampling rate $f_s$=?

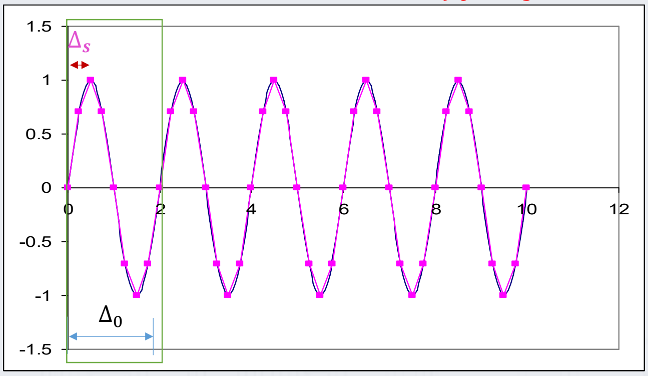
$\Delta_0$**=?** Wave Frequency = ?

Sampling rate = ? X Wave Frequency

## DISCUSSION Q1(E)

$8\Delta_s = 1\Delta_0$ $\quad \frac{f_s}{f_o} = \frac{\Delta_0}{\Delta_s} = 8$

**GOOD SAMPLING**




Sampling time interval Δs =2/8=0.25s

Sampling rate $f_s$=1/ Δs=4 Hz

$\Delta_0$**=2s**, Wave Frequency =1/ $\Delta_0$= 1/2Hz=0.5 Hz

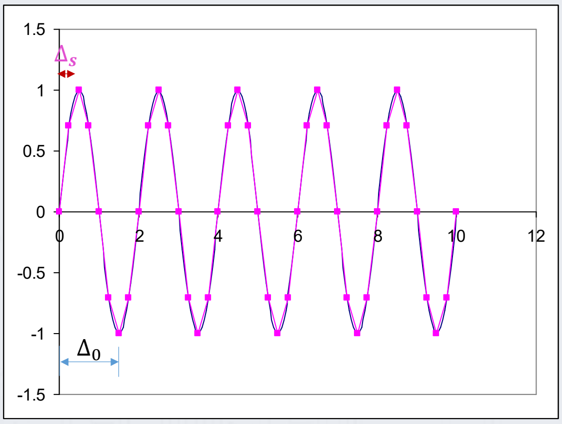
Sampling rate = 8 X Wave Frequency






## DISUCSSION Q1(E)

**GOOD SAMPLING**

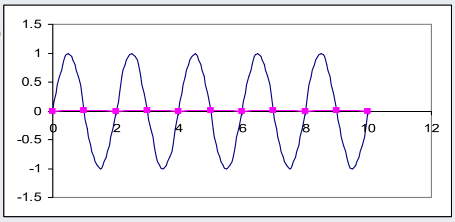


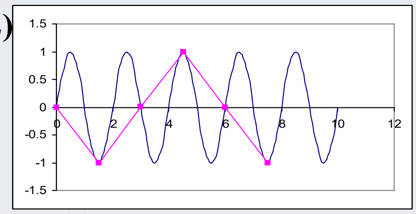

Wave Frequency = 1/2Hz=0.5 Hz

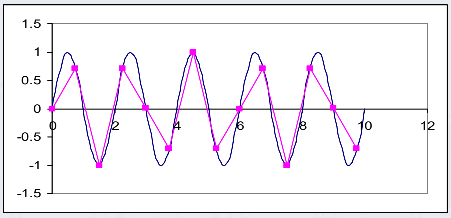
Nyquist Frequency = Wave Frequency=1/2 Hz

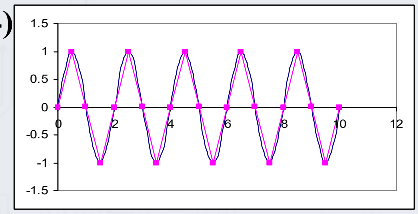
Sampling rate (=4 Hz) > 2* Nyquist Frequency (=1Hz)

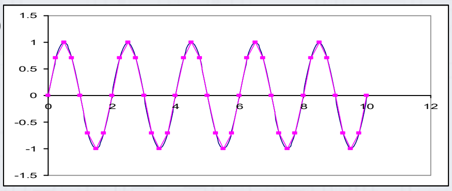
## DISCUSSION Q1(A)

1)



2)



3)



4)



5)



| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Sampling interval | 1 s | 1.5s | 0.75s | 0.5s | 0.25s |
| Wave period/2 | 2/2 s | 2/2 s | 2/2 s | 2/2 s | 2/2 s |

**Sampling interval < Wave period/2?**







## ANSWER TO Q1

**RECOMMENDED SAMPLING**

Sampling rate =10 * Nyquist Frequency

## Q2

$$F(t) = C_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} B_n \sin(n\omega_0 t)$$

Consider the continuous signal:

$$\sum_{n=1} \frac{4}{(2n-1)\pi} sin\left[\frac{2\pi(2n-1)t}{10}\right]$$

What would be an appropriate sampling rate to use in sampling this signal if is to filtered at and above 2 Hz before sampling?

What are the alias frequencies of the filtered signal at this sampling rate?

($f_k$ > 4 Hz; many possible solutions)






## ANSWER TO Q2

The n^th^ term of the series has a frequency $f_n = (2n - 1)/10$.

The fundamental frequency (n = 1) is $f_1 = f_0 = 0.1$ Hz.

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f_n$ = (2n-1)/10 | 1/10 | 1/10 | 3/10 | 5/10 | 7/10 | 9/10 | 11/10 | 13/10 | 15/10 | 17/10 | 19/10 |
| Amplitude(n)= 4/((2n-1)$\pi$) | 4/$\pi$ =1.27 | 4/$\pi$ =1.27 | 4/(3$\pi$) =0.43 | 4/(5$\pi$) =0.25 | 4/(7$\pi$) =0.18 | 4/(9$\pi$) =0.14 | 4/(11$\pi$) =0.12 | 4/(13$\pi$) =0.10 | 4/(15$\pi$) =0.09 | 4/(17$\pi$) =0.08 | 4/(19$\pi$) =0.07 |

# ANSWER TO Q2

The $n^{th}$ term of the series has a frequency $f_n = (2n - 1)/10$.

The fundamental frequency $(n = 1)$ is $f_1 = f_0 = 0.1$ Hz.

$$m = 2n-1, \quad F(t) = \sum_{m=1,3,5,\ldots} \frac{4}{m\pi} sin\left[\frac{2\pi mt}{10}\right], \quad B_m = \begin{cases} \frac{4}{m\pi}, m = 2n-1 \\ 0, m = 2n \end{cases} \quad F(t) = \sum_{m=1}^{\infty} B_m sin\left[\frac{2\pi mt}{10}\right]$$

| n | 0 | | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | | 9 | | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| m | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| $f_m$ = m/10 | | 0 | 1/10 | 2/10 | 3/10 | 4/10 | 5/10 | 6/10 | 7/10 | 8/10 | 9/10 | 10/10 | 11/10 | 12/10 | 13/10 | 14/10 | 15/10 | 16/10 | 18 | 2/10 | 19/10 |
| $f_n$ = (2n-1)/10 | 1/10 | | 1/10 | | 3/10 | | 5/10 | | 7/10 | | 9/10 | | 11/10 | | 13/10 | | 15/10 | | 17/10 | | 19/10 |
| Amplitude $B_m$ | | 0 | 4/π =1.27 | 0 | 4/(3π) =0.43 | 0 | 4/(5π) =0.25 | 0 | 4/(7π) =0.18 | 0 | 4/(9π) =0.14 | 0 | 4/(11π) =0.12 | 0 | 4/(13π) =0.10 | 0 | 4/(15π) =0.09 | 0 | 4/(17π) =0.08 | 0 | 4/(19π) =0.07 |
| Amplitude $B_{2n-1}$= 4/((2n-1)π) | 4/π =1.27 | | 4/π =1.27 | | 4/(3π) =0.43 | | 4/(5π) =0.25 | | 4/(7π) =0.18 | | 4/(9π) =0.14 | | 4/(11π) =0.12 | | 4/(13π) =0.10 | | 4/(15π) =0.09 | | 4/(17π) =0.08 | | 4/(19π) =0.07 |



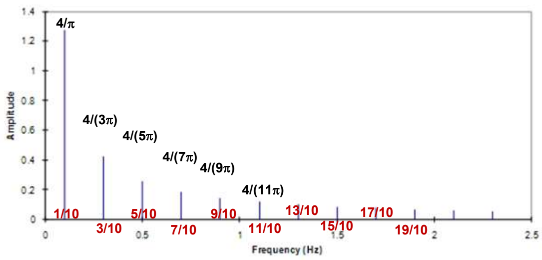




# ANSWER TO Q2

SOLUTION:

This signal has a frequency spectrum as shown:




Filtering the signal at and above 2 Hz limits this series representation of y(t) to 10 terms.

## ANSWER TO Q2

The $n^{th}$ term of the series has a frequency $f_n = (2n-1)/10$.

The fundamental frequency $(n = 1)$ is $f_1 = f_0 = 0.1\ Hz$.

Because $f_{max}$ = 2 Hz, need to set $f_s$ > 4 Hz.

The alias frequencies of the filtered signal depend on the sampling rate. Let us say we select $\boldsymbol{f_s}$ = 5 Hz.



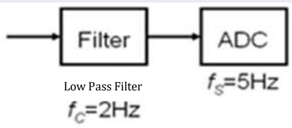


## ANSWER TO Q2

Alias frequencies can be found using

$f_a = \pm f_n + i f_s$ for i = 1 to $\infty$

e.g. $f_a$ = -0.1+5 =4.9 $f_a$ = 0.1+5 =5.1




| n | $f_n$ | i=1 | i=2 | i=3 | i=4 | … |
|---|---|---|---|---|---|---|
| 1 | 0.1 | 4.9, 5.1 | 9.9, 10.1 | 24.9, 15.1 | 19.9, 20.1 | Etc. |
| 2 | 0.3 | 4.7, 5.3 | 9.7, 10.3 | 14.7, 15.3 | 19.7, 20.3 | Etc. |
| 3 | 0.5 | 4.5, 5.5 | 9.5, 10.5 | 14.5, 15.5 | 19.5, 20.5 | Etc. |
| 4 | 0.7 | 4.3, 5.7 | 9.3, 10.7 | Etc. | | |
| 5 | 0.9 | 4.1, 5.9 | 9.1, 10.9 | Etc. | | |
| 6 | 1.1 | 3.9, 6.1 | 8.9, 11.1 | Etc. | | |
| 7 | 1.3 | 3.7, 6.3 | 8.7, 11.3 | Etc. | | |
| 8 | 1.5 | 3.5, 6.5 | Etc. | | | |
| 9 | 1.7 | 3.3, 6.7 | Etc. | | | |
| 10 | 1.9 | 3.1, 6.9 | Etc. | | | |

**Objective function: |i*fs – fn| with variable i=1, 2, 3 ….**

Shaparenko, B. and Cimbala, J. M., Int. J. *Mech. Engr Education*, Vol. 39, No. 3, pp. 195-199, 2012

## ALIAS FREQUENCIES

***Proof of $f_a = i * f_s \pm f_n$, for $i = 1, 2, 3, \ldots$***

Given two analog signals $x(t)$ and $y(t)$:

$$x(t) = A\cos(2\pi f_n t)$$
$$y(t) = A\cos(2\pi f_a t)$$

We sample these two signals at sampling rate of $f_s$:

$$x[n] = A\cos(2\pi f_n \frac{n}{f_s})$$

$$y[n] = A\cos(2\pi f_a \frac{n}{f_s})$$

In the next few steps, we prove that $y[n] = x[n]$, in the case of $f_a = i * f_s - f_n$.

$$y[n] = A\cos\left[2\pi(i * f_s - f_n)\frac{n}{f_s}\right]$$

$$= A\cos\left[2\pi i * f_s * \frac{n}{f_s} - 2\pi f_n \frac{n}{f_s}\right]$$

$$= A\cos[2\pi i * n - 2\pi f_n \frac{n}{f_s}]$$

$$= A\cos(-2\pi f_n \frac{n}{f_s})$$

$$= A\cos(2\pi f_n \frac{n}{f_s})$$

$$= x[n]$$

Same for the case of $f_a = i * f_s + f_n$. Proof is done.







## Q3

a) How many bits are needed to represent the number 756? Without using a calculator, convert 756 to binary

(10 bits; 10 1111 0100)

b) An 8-bit digital-to-analog converter has an output range of 0 to 5V. What is its resolution? Estimate the voltage output if the input code has the decimal value of 32.

(0.0195V; 0.625V)

## ANSWER TO Q3(A)

ANSWER TO Q2

SOLUTION:

$$2^n \geq 756$$

$$n \geq \frac{\log_{10} 756}{\log_{10} 2} = \frac{2.878}{0.301}$$

$$= 9.56 \to 10$$

Thus 10 bits are required.

756 / 2 =378 remainder **0**
378 / 2 = 189 remainder **0**
189 / 2 = 94 remainder **1**
94 / 2 = 47 remainder **0**
47 / 2 = 23 remainder **1**
23 / 2 = 11 remainder **1**
11 / 2 = 5 remainder **1**
5 / 2 = 2 remainder **1**
2 / 2 = 1 remainder **0**
1 / 2 = 0 remainder **1**

Thus $756_{10} = 10\ 1111\ 0100_2$







## ANSWER TO Q3(B)

b) An 8-bit digital-to-analog converter has an output range of 0 to 5V. What is its resolution? Estimate the voltage output if the input code has the decimal value of 32.

(0.0195V; 0.625V)

## ANSWER TO Q3(B)

SOLUTION:

$$\text{Resolution } Q = \frac{(V_{max} - V_{min})}{2^n} = \frac{5}{2^8} = 0.01953125\text{V}$$

An input of 32 will then give an output of

$$V_{out} = 32 \times 0.01953 = 0.625\text{V}$$

COMMENT: Maximum input code is 1111 1111 = 255. Therefore the maximum $V_{out} = 255 \times 0.01953125 = 4.980\text{V}$, not 5V!

Error = 5V - 4.98V ~ **0.01953125V**

Note: $Q=(Vmax-Vmin)/(2^n-1)$ when $2^n$ is fairly large the difference for the results can be ignored both in terms of resolution and output.