NANYANG TECHNOLOGICAL UNIVERSITY

MA2011 MECHATRONICS SYSTEMS INTERFACING

Tutorial 3

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LINEAR SYSTEMS

Linear systems are of the form

$$\sum_{n=0}^{N} A_n \frac{d^n X_{out}}{dt^n} = \sum_{m=0}^{M} B_m \frac{d^m X_{in}}{dt^m}$$

where X_{in} and X_{out} are input and output variables, A_n and B_m are coefficients, N is the order of the system.

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Q1

Which of the following is correct for the Characteristic Equations of the Linear System?

1) $\sum_{n=1}^{N} A_n s^n = 0$

2) $\sum_{n=1}^{N} A_n s^n = 1$

3) $\sum_{n=0}^{N} A_n s^n = 0$

4) $\sum_{n=0}^{N} A_n s^n = 1$

ANSWER TO Q1

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4) $\sum_{n=0}^{N} A_n s^n = 1$

Quiz 3
At
NTULearn

ANSWER TO Q1

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1) $\sum_{n=1}^{N} A_n s^n = 0$

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3)√ $\sum_{n=0}^{N} A_n s^n = 0$

4) $\sum_{n=0}^{N} A_n s^n = 1$

ANSWER TO Q1

Linear equation: $A_1S+A_0=0$

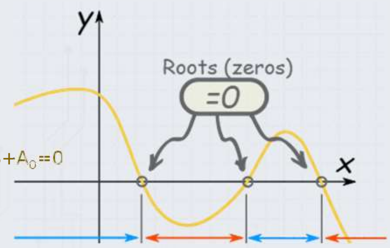
Quadratic equation: $A_2S^2+A_1S+A_0=0$

Cubic equation: $A_3S^3 + A_2S^2+A_1S+A_0=0$

Quartic equation: $CA_2S^2+A_1S+A_0=0$

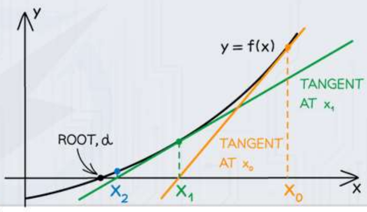
Quintic equation: $A_5S^5 + A_4S^4 + A_3S^3 + A_2S^2+A_1S+A_0=0$

Degree 6 or higher: $\sum_{n=0}^{N} A_n s^n = 0$



NEWTON–RAPHSON FORMULA

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



LINEAR SYSTEMS

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Quiz 3 At NTULearn

where X_{in} and X_{out} are input and output variables, A_n and B_m are coefficients, N is the order of the system.

Homogenous equation of Linear system:

$$\sum_{n=0}^{N} A_n \frac{d^n X_{out}}{dt^n} = 0$$

Characteristic Equation of Homogenous equation of Linear system

$$\sum_{n=0}^{N} A_n S^n = 0$$

T3

Too simple — Zero-order system

Good option — 1st-order system

Too complicated — 2nd-order system

LINEAR SYSTEM

1st Order System

$$\tau \frac{dX_{out}}{dt} + X_{out} = KX_{in}$$

The **Characteristic Equation** of The Homogenous Equation of the 1st Order System

$$\tau s + 1 = 0$$

LINEAR SYSTEM

$$\tau \frac{dX_{out}}{dt} + X_{out} = KX_{in}$$

Step input

Since the root of this equation is s=-1/τ, the **homogeneous solution** is

$$X_{out_h} = Ce^{-t/\tau}$$

Where C is a constant which can determined later applying initial conditions

$$\tau \frac{dCe^{-t/\tau}}{dt} + Ce^{-t/\tau} = \tau * (-\frac{C}{\tau})e^{-t/\tau} + Ce^{-t/\tau} = 0$$

LINEAR SYSTEM

Step input

$$\tau \frac{dX_{out}}{dt} + X_{out} = KX_{in}$$

A **particular solution**

$$X_{out_p} = KA_{in}$$

$$X_{in} = A_{in}$$

$$\tau \frac{dX_{out_p}}{dt} + X_{out_p} = -\tau * 0 + KA_{in} = KA_{in}$$

LINEAR SYSTEM

$$\tau \frac{dX_{out}}{dt} + X_{out} = KX_{in}$$

Step input

The general solution is the sum of the homogeneous and particular solutions

$$X_{out} = X_{out_h} + X_{out_p} = Ce^{-t/\tau} + KA_{in}$$

Q2

3. During a step function calibration, a first-order instrument is exposed to a step change of 100 units. If after 1.2 s the instrument indicates 80 units, estimate the instrument time constant. Estimate the error in the indicated value after 1.5 s. Assume $X_{out}(0)$ = 0 units and K = 1 unit/unit.

τ = 0.75 s; error at 1.5 s = 13.4 units

LINEAR SYSTEM

Determining constant C by applying initial conditions

$$X_{out}(0) = 0$$

$$X_{out}(0) = Ce^{-0/\tau} + KA_{in} = 0$$

$$C = -KA_{in}$$

$$X_{out}(t) = Ce^{-t/\tau} + KA_{in} = KA_{in}(1 - e^{-t/\tau})$$

Q2

3. During a step function calibration, a first-order instrument is exposed to a step change of 100 units. If after 1.2 s the instrument indicates 80 units, estimate the instrument time constant. Estimate the error in the indicated value after 1.5 s. Assume $X_{out}(0)$ = 0 units and K = 1 unit/unit.

τ = 0.75 s; error at 1.5 s = 13.4 units

$$K = 1$$

$$\lim_{t\to\infty} X_{out}(t) = 100 = 1 * A_{in}\left(1 - \lim_{t\to\infty} e^{-t/\tau}\right)$$

$$A_{in} = 100$$

ANSWER TO Q2

3. During a step function calibration, a first-order instrument is exposed to a step change of 100 units. If after 1.2 s the instrument indicates 80 units, estimate the instrument time constant. Estimate the error in the indicated value after 1.5 s. Assume $X_{out}(0)$ = 0 units and K = 1 unit/unit.

τ = 0.75 s; error at 1.5 s = 13.4 units

$$X_{out}(t) = 100(1 - e^{-t/\tau})$$

$$X_{out}(1.2) = 80 = 100(1 - e^{-1.2/\tau})$$

Thus

$$e^{-1.2/\tau} = 20/100$$

$$-1.2/\tau = \ln(0.2) = -1.61$$

$$\tau \approx 0.75$$

$$X_{out}(t) = 100(1 - e^{-t/0.75})$$

ANSWER TO Q2

3. During a step function calibration, a first-order instrument is exposed to a step change of 100 units. If after 1.2 s the instrument indicates 80 units, estimate the instrument time constant. Estimate the error in the indicated value after 1.5 s. Assume $X_{out}(0)$ = 0 units and K = 1 unit/unit.

τ = 0.75 s; error at 1.5 s = 13.4 units

$$X_{out}(t) = 100(1 - e^{-t/0.75})$$

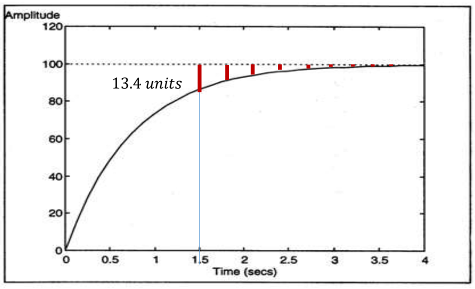
$$X_{out}(1.5) = 100(1 - e^{-1.5/0.75})$$

Thus

error at 1.5 s =100- $X_{out}(1.5)$ =$100e^{-1.5/0.75} = 13.4\ units$

ANSWER TO Q2

error at t =100- $X_{out}(t)$ =$100e^{-t/0.75}$



Step response of system

Note that two parameters, i.e. τ and K, are needed to characterise the first order system. τ and K are system variables, and are **not** dependent on the input.

Q3

A first-order instrument with a time constant of 2 s is to be used to measure a periodic input. If a dynamic error of 2% can be tolerated, determine the maximum frequency of a periodic input that can be measured.

($\omega_{max} = 0.1$ rad s^{-1})

$$\tau = 2$$

ANSWER TO Q3

1. **Magnitude Ratio** can be considered as Attenuation (always positive and less than 1)

$$M(\omega) = 1/\sqrt{1 + (\omega\tau)^2}$$

2. **Dynamic error** (1st order system) always positive and less than 1 as well

$$\delta(\omega) = 1 - M(\omega)$$

ANSWER TO Q3

A first-order instrument with a time constant of 2 s is to be used to measure a periodic input. If a dynamic error of 2% can be tolerated, determine the maximum frequency of a periodic input that can be measured.

($\omega_{max} = 0.1$ rad s^{-1})

Solution

$$\delta(\omega) = 1 - M(\omega) \le 0.02$$

$$0.98 \le M(\omega) \le 1$$

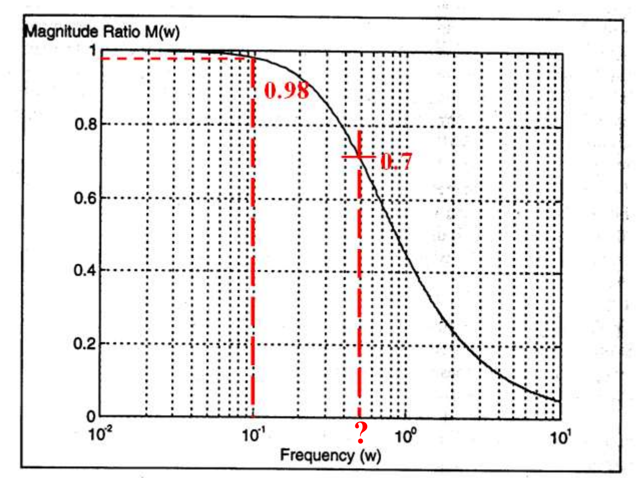
$$0.98 \le 1/\sqrt{1 + (\omega\tau)^2} \le 1$$

$$0 \le \omega \le \sqrt{(1/0.98)^2 - 1}/\tau$$

τ=2, ω_{max} =0.1 rad s^{-1} or f_{max} =0.016 Hz

ANSWER TO Q3

$$M(\omega) = 1/\sqrt{1 + (\omega\tau)^2}$$

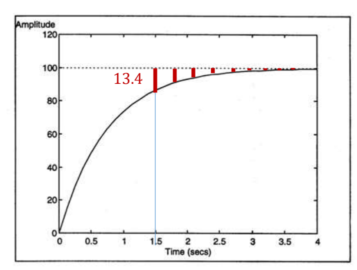


Magnitude ratio plot of system

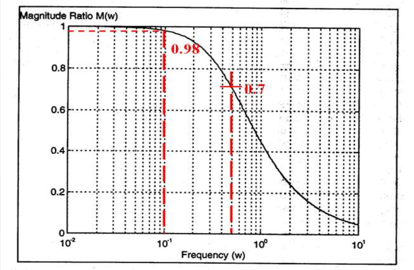
ANSWER TO Q3

The first example Q2 looks at the time response of a system, i.e. in the time domain.

The second example Q3 looks at its frequency response, i.e. in the frequency domain. Both are often needed, especially for a more complicated system, to have a good understanding of the system.



Step response of system



Magnitude ratio plot of system

Q4

The output from a temperature system indicates a steady, time-varying signal having amplitude which varies between 30 and 40 °C, with a single frequency of 10 Hz. Express the output signal as a waveform equation, $T(t)$. If the dynamic error is to be less than 1%, what must the system time constant be? Assume that the sensitivity $K = 1$ and system is of first order.

($T(t) = 35 + 5\sin(20\pi t \pm \phi)$; $\tau \leq 2.27$ ms)

$$f = 10 \text{ Hz}$$

1st order system

ANSWER TO Q4

f =10 Hz for one single sine wave

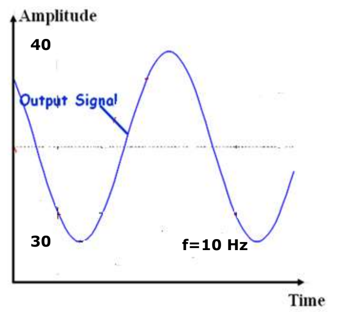
SOLUTION:

Amplitude varies between 30°C and 40°C, and frequency is 10 Hz.

Assume sensitivity K = 1 and system is first order.

Output signal is $\boxed{T(t) = \frac{40+30}{2} + \frac{40-30}{2}\sin(2\pi ft \pm \phi)}$

$= 35 + 5\sin(20\pi t \pm \phi)$



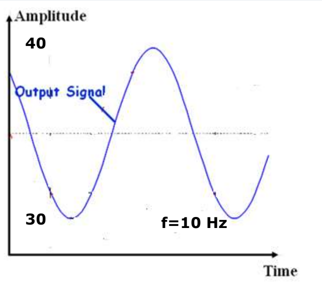
ANSWER TO Q4

$= 35 + 5\sin(20\pi t \pm \phi)$

ϕ is the unknown phase shift between input and output signal. For a 1st order system, the maximum phase shift is $\pi/2$ radians.

Dynamic error is $\delta(\omega) = 1 - M(\omega)$ and $M(\omega) = \frac{1}{\sqrt{1+(\omega\tau)^2}}$

And we know that $\delta \leq 0.01$, hence



ANSWER TO Q4

$$1 \geq \frac{1}{\sqrt{1+(\omega\tau)^2}} \geq 0.99$$

At $\omega = 2\pi f = 20\pi$ rad s^{-1}, we find that

$$0 \leq \tau \leq 2.27 \text{ ms}$$